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## Invariant manifolds for a singular ordinary differential equation

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$$\frac{dU}{dt} = f(U)/z(U) + g(U),$$

where  $U \in R^N$ , the functions  $f \in R^N$  and  $g \in R^N$  are of class  $C^2$  and  $z$  is a real valued  $C^2$  function. The equation is singular in the sense that  $z(U)$  can attain the value 0. We focus on the solutions of the singular ODE that belong to a small neighborhood of a point  $\bar{U}$  such that  $f(\bar{U}) = g(\bar{U}) = \vec{0}$ ,  $z(\bar{U}) = 0$ . We investigate the existence of manifolds that are locally invariant for the singular ODE and that contain orbits with a suitable prescribed asymptotic behaviour. Under suitable hypotheses on the set  $\{U : z(U) = 0\}$ , we extend to the case of the singular ODE the definitions of center manifold, center stable manifold and of uniformly stable manifold. An application of our analysis concerns the study of the viscous profiles with small total variation for a class of mixed hyperbolic-parabolic systems in one space variable. Such a class includes the compressible Navier Stokes equation.

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# Invariant manifolds for a singular ordinary differential equation

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## Abstract

We study the singular ordinary differential equation

$$\frac{dU}{dt} = \frac{1}{\zeta(U)} \phi_s(U) + \phi_{ns}(U), \quad (0.1)$$

where  $U \in \mathbb{R}^N$ , the functions  $\phi_s \in \mathbb{R}^N$  and  $\phi_{ns} \in \mathbb{R}^N$  are smooth and  $\zeta$  is a real valued regular function. The equation is singular in the sense that  $\zeta(U)$  can attain the value 0. We focus on the solutions of (0.1) that belong to a small neighbourhood of a point  $\bar{U}$  such that  $\phi_s(\bar{U}) = \phi_{ns}(\bar{U}) = \vec{0}$ ,  $\zeta(\bar{U}) = 0$ . We prove the existence of two invariant manifolds for (0.1), which can be regarded respectively as a centre and a generalized stable manifold. Also, we prove a decomposition result for the generalized stable manifold.

An application of our analysis concerns the study of the viscous profiles with small total variation for mixed hyperbolic-parabolic systems in one space variable. We consider explicitly the case of the compressible Navier Stokes equation.

## 1 Introduction

In this work we study the singular ordinary differential equation

$$\frac{dU}{dt} = \frac{1}{\zeta(U)} \phi_s(U) + \phi_{ns}(U). \quad (1.1)$$

In the previous expression,  $U \in \mathbb{R}^N$  and the functions  $\phi_s$  and  $\phi_{ns}$  are smooth and take value into  $\mathbb{R}^N$ . The function  $\zeta$  is as well regular and it takes real values. We say that the equation is singular because  $\zeta(U)$  can attain the value 0.

The following system is widely studied in singular perturbation theory:

$$\begin{cases} \varepsilon d\mathbf{x}/dt = f(\mathbf{x}, \mathbf{y}, \varepsilon) \\ d\mathbf{y}/dt = g(\mathbf{x}, \mathbf{y}, \varepsilon) \end{cases} \quad (1.2)$$

Here  $\mathbf{x}$  and  $\mathbf{y}$  are vector valued functions,  $\varepsilon$  is a parameter and one typically studies the limit  $\varepsilon \rightarrow 0$ . The literature concerning this topic is very rich, so it would be difficult to give an overview of the results concerning (1.2) and the several related problems: here we just refer to [11, 10] and to the bibliography contained therein. Note that (1.2) can be written in the form (1.1): in this case, the singularity  $\zeta$  is  $\varepsilon$  and hence it is a parameter,  $d\zeta/dt = 0$ .

The main novelty of this work is that we consider the case the  $\zeta$  is a function of the unknown  $U$ . In particular, this means that  $d\zeta/dt \neq 0$  in general and hence that we have to face the possibility that  $\zeta(U(0)) \neq 0$ , but  $\zeta(U(t)) = 0$  for a finite value of  $t$ . This is exactly what happens in the examples (2.11) and (2.16) discussed in Section 2. Other examples are provided in [4], Section 2. Note that, in all these cases, there is a loss of regularity at the time  $t_0$  at which  $\zeta(U(t))$  reaches the value 0,  $t_0 = \min \{t \in [0, +\infty[ : \zeta(U(t)) = 0\}$ . More precisely, the first derivative  $dU/dt$  either has a discontinuity or blows up at  $t = t_0$ .

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Our goal here is to study the solutions of (1.1) that lay in a neighborhood of a point  $\bar{U}$  such that  $\phi_s(\bar{U}) = \phi_{ns}(\bar{U}) = \bar{0}$  and  $\zeta(\bar{U}) = 0$ . We prove that, under suitable hypotheses, one can define manifolds that are invariant for (1.1) and that satisfy the following property: if  $U$  is an orbit laying on the manifold and  $\zeta(U(0)) \neq 0$ , then  $\zeta(U(t)) \neq 0$  for every  $t$ . We proceed as follows.

First, we define the *manifold of slow dynamics* and the *manifold of fast dynamic* of (1.1). These can be regarded as extensions to the general case of the notions of *slow* and *fast time scale* discussed for example in [8] in relation to system (1.2), namely to the case  $\zeta$  is a parameter. The manifold of slow dynamics is invariant for (1.1) and every orbits satisfies the property we mentioned before: if  $\zeta(U(0)) \neq 0$ , then  $\zeta(U(t)) \neq 0$  for every  $t$ . Moreover, the centre manifold of (1.1) is contained in the manifold of slow dynamics (Theorem 4.1).

We also study the solutions of (1.1) that decay exponentially to an equilibrium when  $t \rightarrow +\infty$ . The related results are collected in Theorem 4.2. More precisely, we proceed as follows. We fix a manifold of equilibria  $E$  which contains the point  $\bar{U}$  and whose exact definition can be found in Section 4. We show that there exists a manifold  $\mathcal{M}^s$ , invariant for (1.1), such that every orbit laying on  $\mathcal{M}^s$  converges exponentially fast to a point in  $E$ . Note that  $\mathcal{M}^s$  is not the stable manifold because we do not require that the orbits converge to the given point  $\bar{U}$ . Conversely, we ask that every orbit on  $\mathcal{M}^s$  decays to a limit which is not prescribed a priori, but we only know belongs to  $E$ . In the following, we refer to  $\mathcal{M}^s$  as to the *generalized stable manifold*. We also prove a decomposition result: every orbit laying on  $\mathcal{M}^s$  can be written as the sum of an orbit laying on the manifold of slow dynamics, an orbit laying on the manifold of fast dynamics and an interaction term, which is small with respect to the other two, in the sense specified in Section 4. As a consequence, we get that the generalized stable manifold, too, satisfies the property we mentioned before: if  $U$  lays on  $\mathcal{M}^s$  and  $\zeta(U(0)) \neq 0$ , then  $\zeta(U(t)) \neq 0$  for every  $t$ .

The hypotheses assumed in the work are discussed in Section 2. In particular, Hypothesis 6 guarantees that the set  $\{U : \zeta(U) = 0\}$  is invariant with respect to the manifold of the fast dynamics. Example (2.11) shows that, if all the conditions listed in Section 2 but Hypothesis 6 are satisfied, then there might be a solution  $U$  of (1.1) which experiences the loss of regularity described before. Namely,  $\zeta(U(0)) \neq 0$ , but  $\zeta(U(t_0)) = 0$  for a finite value  $t_0$ . The first derivative  $dU/dt$  blows up when  $t \rightarrow t_0^-$ . Hypothesis 7, instead, ensures that the set  $\{U : \zeta(U) = 0\}$  is invariant with respect the manifold of slow dynamics. Example (2.16) satisfies all the conditions listed in Section 2 except for Hypothesis 7: as before, we find a solution  $U$  of (1.1) that has loss of regularity and blow up in the first derivative.

An application of our analysis concerns the study of the viscous profiles with small total variation for a mixed hyperbolic-parabolic system in one space dimension. The connection between these viscous profiles and the singular ordinary differential equation (1.1) is discussed in [5], where we also explain what we mean by viscous profiles and by mixed hyperbolic-parabolic systems in this context. Here we just point out that these viscous profiles are solutions of particular ordinary differential equations.

We refer to the books [6, 7, 14] and to the rich bibliography contained therein for a discussion about the applications of viscous profiles to the study of conservation laws. Also, concerning the issue of stability, we refer to [15], to [3] and to the references therein.

In [4] we introduced a new condition, the so called block linear degeneracy, and we used it to study the viscous profiles. However, as it was pointed out by Frédéric Rousset in [12], the condition of block linear degeneracy is satisfied by the compressible Navier Stokes equation written in Lagrangian coordinates, but is violated when the same equation is written in Eulerian coordinates. The problem is that the viscous profiles for the Navier Stokes equation written in Eulerian coordinates satisfy a singular equation in the form (1.1). In [5] we show that, to extend the analysis in [4], one has to find conditions, satisfied by the Navier Stokes equation written in Eulerian coordinates, which guarantee the following. If  $U$  is a solution of (1.1) that lay either on a centre or on the generalized stable manifold and satisfies  $\zeta(U(0)) \neq 0$ , then  $\zeta(U(t)) \neq 0$  for every  $t$ . In Section 2.3 we show that our hypotheses are all satisfied by the viscous profiles of the Navier Stokes written in Eulerian coordinates and thus they are the conditions we were looking for.

Note that in [4] the analysis of the viscous profiles for a mixed hyperbolic-parabolic system was used to study a family of initial boundary value problems defined by the partial differential equation

$$E(u)u_t + A(u, u_x)u_x = B(u)u_{xx}. \quad (1.3)$$

Here the function  $u$  takes values into  $\mathbb{R}^d$  and depends on the two scalar variables  $t$  and  $x$ . In [4] both cases are contemplated: *characteristic* and *non characteristic boundary*. The boundary is non characteristic for

(1.3) if none of the eigenvalues of  $A$  can attain the value 0. In the case of a *characteristic boundary* one of the eigenvalues of  $A$  can indeed attain the value 0. For technical reasons, the results described in this work can be directly applied only to the case of a non characteristic boundary. Indeed, to study the characteristic case one needs to handle viscous profiles that lay neither on the centre nor on the generalized stable manifold.

The exposition is organized as follows. In Section 1.1 we discuss a simplified example and we introduce the results that are extended in the following sections to the general case.

In Section 2 we define our hypotheses and in Section 2.3 we show that they are satisfied by the viscous profiles of the compressible Navier Stokes equation. Also, in Section 2.2 we discuss two examples: each of them show that, if one different hypothesis is not satisfied, then the first derivative  $dU/dt$  of a solution of (1.1) may blow up in finite time.

In Section 3 we define a change of coordinates that reduces (1.1) to a more convenient form. Finally, in Section 4 we discuss our main results, which are Theorems 4.1 and 4.2. In Section 4.1 we introduce the notion of fast and slow dynamics and we show that a centre manifold contains only slow dynamics. Section 4.2 concerns the generalized stable manifold and we prove that every orbit laying on the manifold can be decomposed as the sum of a slow dynamic, a fast dynamics and a small interaction term.

## 1.1 A toy model

In this section we discuss a toy model for system (1.1). In this way, we introduce in a simplified context the analysis that is extended in Section 4.1 to the general case. All the hypotheses stated in Section 2 are satisfied by the toy model discussed here.

Also, note that in this specific case one could actually apply the tools described for example in [11]. Indeed, we suppose that in (1.1)  $\zeta$  is just a parameter, namely

$$\frac{d\zeta}{dt} = 0. \quad (1.4)$$

Also, we assume that the function  $V(t) \in \mathbb{R}^3$  satisfies

$$\frac{dV}{dt} = \frac{1}{\zeta} A_s V + A_{ns} V, \quad (1.5)$$

where

$$A_s = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_{ns} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote by  $V = (v_1, v_2, v_3)^T$  the components of  $V$  and let

$$U = \begin{pmatrix} \zeta \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \phi_s(U) = \begin{pmatrix} 0 \\ -3v_1 \\ 0 \\ 0 \end{pmatrix} \quad \phi_{ns}(U) = \begin{pmatrix} 0 \\ 0 \\ -2v_2 \\ v_3 \end{pmatrix}, \quad (1.6)$$

then (1.4) and (1.5) can be written as

$$\frac{dU}{dt} = \frac{1}{\zeta} \phi_s(U) + \phi_{ns}(U). \quad (1.7)$$

In the following we consider only non negative values of  $t$  and we focus on the limit  $\zeta \rightarrow 0^+$ . The study of the limit  $\zeta \rightarrow 0^-$  is completely analogous.

In this case, we have an explicit solution of (1.7):

$$\begin{cases} \zeta(t) = \zeta(0) \\ v_1(t) = v_1(0)e^{-3t/\zeta(0)} \\ v_2(t) = v_2(0)e^{-2t} \\ v_3(t) = v_3(0)e^t \end{cases}$$

The component  $v_1(t)$  can be regarded as a *singular* or *fast* dynamic. Indeed, the speed of exponential decay of  $v_1$  gets faster and faster as  $\zeta(0) \rightarrow 0^+$ . Also,  $v_1$  is singular for  $\zeta(0) \rightarrow 0^+$  in the sense that it converges pointwise to a jump, namely

$$\lim_{\zeta(0) \rightarrow 0^+} v_1(0)e^{-3t/\zeta(0)} = \begin{cases} v_1(0) & \text{if } t = 0 \\ 0 & \text{if } t > 0 \end{cases}$$

On the other side,  $\zeta(t)$ ,  $v_2(t)$  and  $v_3(t)$  can be regarded as *non singular* or *slow* dynamics. Indeed, when  $\zeta(0) \rightarrow 0^+$  they do not show any singular behaviour. Also, as  $\zeta(0) \rightarrow 0^+$  the speed of exponential decay of  $v_2$  is slow if compared to the speed of exponential decay of  $v_1(t)$ .

In Section 4 we give an explicit definition of *slow* and *fast* dynamics, which is valid in the general case where the system is non linear and  $d\zeta/dt \neq 0$ .

Here, instead, we point out the following. Let

$$V^{--} := \left\{ (0, v_1, 0, 0) : v_1 \in \mathbb{R} \right\} \quad V^{0-} := \left\{ (0, 0, v_2, 0) : v_2 \in \mathbb{R} \right\}.$$

Also, consider the following subspace, which is entirely constituted by equilibria:

$$E := \left\{ (\zeta, 0, 0, 0) : \zeta \in \mathbb{R} \right\}.$$

If we set

$$\mathcal{M}^s := V^{--} \oplus V^{0-} \oplus E,$$

then  $\mathcal{M}^s$  may be regarded as a *generalized stable manifold* in the sense that, given a point  $(\bar{\zeta}, \bar{v}_1, \bar{v}_2, 0)$  belonging to  $\mathcal{M}^s$ , then the solution of (1.7) starting at  $(\bar{\zeta}, \bar{v}_1, \bar{v}_2, 0)$  is

$$\begin{cases} \zeta(t) = \bar{\zeta} \\ v_1(t) = \bar{v}_1 e^{-3t/\bar{\zeta}} \\ v_2(t) = \bar{v}_2 e^{-2t} \\ v_3(t) = 0. \end{cases} \quad (1.8)$$

Thus, the orbit starting at  $(\bar{\zeta}, \bar{v}_1, \bar{v}_2, 0)$  decays exponentially to a point of  $E$ , namely to  $(\bar{\zeta}, 0, 0, 0)$ .

Note that (1.8) can be decomposed as

$$\begin{pmatrix} \bar{\zeta} \\ 0 \\ \bar{v}_2 e^{-2t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \bar{v}_1 e^{-3t/\bar{\zeta}} \\ 0 \end{pmatrix}, \quad (1.9)$$

namely it is the sum of a *slow* and a *fast* dynamic. The slow dynamic is the first term in the previous expression, the fast dynamic the second one. In Section 4.2 we introduce a definition of *generalized stable manifold* valid for a general singular non linear equation (1.1). Also, we prove a decomposition result similar to (1.9). Namely, we show that every orbit laying on the generalized stable manifold can be written as the sum of a slow dynamic, a fast dynamic and an interaction component. The interaction component does not appear in (1.9), but it does in the general case. However, it is small (in the sense explained in Section 4.2) with respect to the slow and the fast dynamics.

## 2 Hypotheses

In this section we discuss the hypotheses assumed in the work.

More precisely, in Section 2.1 we state the conditions imposed on the singular O.D.E.

$$\frac{dU}{dt} = \frac{1}{\zeta(U)} \phi_s(U) + \phi_{ns}(U). \quad (2.1)$$

In Section 2.2 we discuss two counterexamples. They show that, if some of the conditions introduced in Section 2.1 are not satisfied, then there might be solutions of (2.1) that are not continuously differentiable.

Finally, in Section 2.3 we verify that the conditions introduced in Section 2.1 are satisfied by the viscous profiles of the compressible Navier Stokes equation written in Eulerian coordinates.

## 2.1 Hypotheses satisfied by the singular O.D.E

Define

$$F(U) = \phi_s(U) + \zeta(U)\phi_{ns}(U) \quad (2.2)$$

and consider the non singular ordinary differential equation

$$\frac{dU}{d\tau} = F(U) \quad U \in \mathbb{R}^N. \quad (2.3)$$

Formally, (2.3) is obtained from (2.1) *via* the change of variables  $\tau = \tau(t)$  defined as the solution of the Cauchy problem

$$\begin{cases} \frac{d\tau}{dt} = \frac{1}{\zeta[U(t)]} \\ \tau(0) = 0. \end{cases}$$

However, the function  $\tau(t)$  is well defined only if  $\zeta[U(t)] \neq 0$  for every  $t$ . To overcome this difficulty we then proceed as follows: we state all the hypotheses referring to the formulation (2.3). Relying on these hypotheses, in Section 4 we prove the existence of two invariant manifolds for (2.3) which satisfy the following property. If  $U$  is an orbit laying on one of these manifolds and  $\zeta[U(0)] \neq 0$ , then  $\zeta[U(t)] \neq 0$  for every  $t$ . If we restrict to the orbits laying on these manifolds, (2.3) is equivalent to (2.1). Also, as pointed out in the introduction, the viscous profiles with small total variation always do lay on one of these manifolds.

Before stating the hypotheses, we recall that we want to study (2.1) and (2.3) in the neighbourhood of an equilibrium point  $\vec{U}$  such that  $F(\vec{U}) = \vec{0}$  and  $\zeta(\vec{U}) = 0$ . It is not restrictive to take  $\vec{U} = \vec{0}$ . Namely, in the following we assume

$$F(\vec{0}) = \vec{0} \quad \zeta(\vec{0}) = 0. \quad (2.4)$$

Also, it is not restrictive to assume that all the eigenvalues of  $DF(\vec{0})$  have non positive real part. Indeed, this condition is satisfied if we restrict to a centre-stable manifold for (2.3). Note that the viscous profiles with small total variation of mixed hyperbolic-parabolic systems always belong to every centre-stable manifold. We refer to [9] for an extensive discussion about centre-stable manifolds, here we just recall the most important properties.

Consider the general case:  $DF(\vec{0})$  has eigenvalues with negative, positive and zero real part. Denote by  $V^c$  the eigenspace associated with eigenvalues with null real part and denote by  $V^s$  the eigenspace associated to eigenvalues with strictly negative real part. Also, let  $c > 0$  such that, if  $\lambda$  is an eigenvalue of  $DF(\vec{0})$  with strictly positive real part, then  $Re(\lambda) > c$ . There exists a so called centre-stable manifold, namely a manifold which is defined in a sufficiently small neighborhood of  $\vec{0}$ , is parameterized by the direct sum  $V^c \oplus V^s$ , is locally invariant for the flux of (2.3) and satisfies the following property. If  $U_0$  belongs to such a manifold, then the solution of the Cauchy problem

$$\begin{cases} \frac{dU}{dt} = F(U) \\ U(0) = U_0 \end{cases}$$

satisfies

$$\lim_{x \rightarrow +\infty} |U(x)|e^{-cx/2} = 0.$$

Fix a centre manifold manifold  $\mathcal{M}^{cs}$  for (2.3), defined in a neighbourhood of  $\vec{0}$ .

To study the solutions of (2.3) laying on  $\mathcal{M}^{cs}$  one can proceed as described for example in [4], Section 3.2.3. Here we just recall the most important steps. For every  $\tau$  we can write

$$U(\tau) = R_{cs}(V^{cs}(\tau))V_{cs}(\tau),$$

where  $V_{cs} \in V^c \oplus V^s$  and  $R_s$  is a suitable matrix belonging to  $\mathbb{M}^{N \times n_{cs}}$ . Here  $N$  is the dimension of  $U$  and  $n_{cs}$  is equal to the sum between the dimension of  $V^c$  and the dimension of  $V^s$ . Also, one can show that  $V^{cs}$  satisfies

$$\frac{dV^{cs}}{d\tau} = F^{cs}(V^{cs}), \quad (2.5)$$

for a suitable function  $F^{cs}$  such that the jacobian  $DF^{cs}(\vec{0})$  admits only eigenvalues with non positive real part.

Since the solutions we are interested in (viscous profiles with small total variation) lay on every centre-stable manifold, we can study directly system (2.5). To simplify notations in the following we use the formulation (2.3), but we assume that Hypothesis 1 holds.

**Hypothesis 1.** *The jacobian  $DF(\vec{0})$  admits only eigenvalues with non positive real part.*

To simplify the exposition, we also assume

**Hypothesis 2.** *The initial datum  $U(0)$  of (2.3) is such that  $\zeta(U(0)) \geq 0$ .*

The case  $\zeta(U(0)) < 0$  can be handled with the same techniques discussed here.

Also, we assume the following.

**Hypothesis 3.** *The gradient  $\nabla\zeta(\vec{0}) \neq \vec{0}$ .*

Let  $\mathcal{S}$  be the singular set

$$\mathcal{S} := \{U : \zeta(U) = 0\}. \quad (2.6)$$

Thanks to the implicit function theorem, Hypothesis 3 ensures that in a small enough neighbourhood of  $\vec{0}$  the set  $\mathcal{S}$  is actually a manifold of dimension  $N - 1$ , where  $N$  is the dimension of  $U$ .

**Hypothesis 4.** *Let  $\mathcal{M}^c$  be any centre manifold for (2.3) around the equilibrium point  $\vec{0}$ . There exists a positive constant  $\delta \ll 1$  such that if  $|U| \leq \delta$  and  $U$  belongs to the intersection  $\mathcal{M}^c \cap \mathcal{S}$ , then  $U$  is an equilibrium for (2.3), namely  $F(U) = \vec{0}$ .*

Concerning equilibria, we also assume the following

**Hypothesis 5.** *There exists a manifold  $\mathcal{M}^{eq}$  of equilibria for (2.3) which contains  $\vec{0}$  and which is transversal to  $\mathcal{S}$ .*

Let  $n_{eq}$  be the dimension of  $\mathcal{M}^{eq}$ . We recall that the manifolds  $\mathcal{S}$  and  $\mathcal{M}^{eq}$  are transversal if the intersection  $\mathcal{S} \cap \mathcal{M}^{eq}$  is a manifold with dimension  $n_{eq} - 1$  (as pointed out before, the dimension of  $\mathcal{S}$  is  $N - 1$ ).

**Hypothesis 6.** *For every  $U \in \mathcal{S}$ ,*

$$\nabla\zeta(U) \cdot F(U) = 0. \quad (2.7)$$

**Hypothesis 7.** *Let  $U \in \mathcal{S}$  be an equilibrium for (2.3), namely  $\zeta(U) = 0$  and  $F(U) = \vec{0}$ . Then*

$$\nabla\zeta(U)DF(U)\left(\nabla\zeta(U)\right)^T = 0. \quad (2.8)$$

In Section 1.1 we introduced, in the case of a toy model, the notion of *slow* and *fast* dynamics. These notions are extended in Section 4.1 to the general non linear case. Hypotheses 7 and 6 can be then reformulated saying that the set  $\mathcal{S}$  is invariant for respectively the manifold of the slow and of the fast dynamics.

**Remark** Consider system

$$\frac{dU}{dt} = \frac{1}{\zeta(U)}\phi_s(U) + \phi_{ns}(U). \quad (2.9)$$

Also, let  $f(U)$  be a regular, real valued function such that  $f(\vec{0}) > 0$ . Clearly, (2.9) is equivalent to

$$\frac{dU}{dt} = \frac{1}{\zeta(U)f(U)}\phi_s(U)f(U) + \phi_{ns}(U). \quad (2.10)$$

and  $\zeta(U)f(U) \rightarrow 0^+$  if and only if  $\zeta(U) \rightarrow 0^+$ , at least in a sufficiently small neighbourhood of  $U = \vec{0}$ . By direct check, one can verify that Hypotheses 1 ... 7 are verified by the couple  $(\zeta, F)$  if and only if they are verified by the couple  $(\zeta f, Ff)$ .

This is important because when we are given a singular ordinary differential equation (2.9) and (2.10) are completely equivalent and hence one has infinitely many choices to define the singularity  $\zeta(U)$ .

## 2.2 Examples

### 2.2.1 Example (2.11)

Example (2.11) deals with a system which satisfies Hypotheses 1 ... 5 and Hypothesis 7, but does not satisfy Hypothesis 6. We exhibit a solution of this system which has a blow up in the first derivative and hence it is not continuously differentiable. As pointed out in [5], this might be regarded as a pathological behaviour if the goal is to apply the analysis to the viscous profiles of mixed hyperbolic-parabolic systems. The loss of regularity experienced in Example (2.11) regards a solution  $U$  such that  $\zeta[U(0)] \neq 0$ , but  $\zeta(U)$  reaches the value 0 for a finite value of  $t$ .

Consider the following system:

$$\begin{cases} du_1/dt = -u_2/u_1 \\ du_2/dt = -u_2 \end{cases} \quad (2.11)$$

Let  $U = (u_1, u_2)^T$ ,  $\zeta(U) = u_1$  and

$$\phi_s(U) = \begin{pmatrix} -u_2 \\ 0 \end{pmatrix} \quad \phi_{ns}(U) = \begin{pmatrix} 0 \\ -u_2 \end{pmatrix}.$$

System (2.11) can then be written in the form

$$\frac{dU}{dt} = \frac{1}{\zeta(U)} \phi_s(U) + \phi_{ns}(U).$$

In this case, the function  $F(U)$  defined by (2.2) is

$$F(U) = \begin{pmatrix} -u_2 \\ -u_2 u_1 \end{pmatrix}.$$

By direct check, one can verify that Hypotheses 1 ... 5 and Hypothesis 7 are satisfied by (2.11). On the other side, Hypothesis 6 is not verified in this case. Indeed, the singular surface  $\mathcal{S}$  defined by 2.6 is in this case the line  $\{u_1 = 0\}$  and

$$\nabla \zeta \cdot F = -u_2$$

is in general different from 0 on  $\mathcal{S}$ .

The solution of (2.11) can be explicitly computed and it is given by

$$\begin{cases} u_1(t) = \sqrt{u_1(0) + u_2(0)(e^{-t} - 1)} \\ u_2(t) = u_2(0)e^{-t} \end{cases} \quad (2.12)$$

Choosing  $u_2(0) > 0$ ,  $u_1(0) > 0$  and small enough, one has that the solution  $u_1(t)$  can reach the value 0 for a finite  $t$ . Note that at that point  $t$  the first derivative  $du_1/dt$  blows up: thus, the solution (2.12) of (2.11) is not  $\mathcal{C}^1$ .

### 2.2.2 Example (2.13)

Example (2.2.2) deals with system (2.13), which is apparently very similar to (2.11). However, in the case of (2.13) Hypotheses 1 ... 7 are all verified. We show the solutions of (2.13) are regular. Also, if  $\zeta[U(0)] \neq 0$  then  $\zeta[U(t)] \neq 0$  for all values of  $t$ .

Consider system

$$\begin{cases} du_1/dt = -u_2 \\ du_2/dt = -u_2/u_1 \end{cases} \quad (2.13)$$

Set  $U = (u_1, u_2)^T$ ,  $\zeta(U) = u_1$ . System (2.13) can be written in the form

$$\frac{dU}{dt} = \frac{1}{\zeta(U)} \phi_s(U) + \phi_{ns}(U)$$



provided that

$$\phi_s(U) = \begin{pmatrix} 0 \\ -u_2 \end{pmatrix} \quad \phi_{ns}(U) = \begin{pmatrix} -u_2 \\ 0 \end{pmatrix}.$$

Also, the function  $F(U)$  defined by (2.2) is in this case

$$F(U) = \begin{pmatrix} -u_2 u_1 \\ -u_2 \end{pmatrix}.$$

By direct check, one can verify that Hypotheses 1 ... 7 are all verified in this case.

To study system (2.13) we can proceed as follows. From (2.13) we have

$$\frac{du_1/dt}{u_1} = -\frac{u_2}{u_1} = du_2/dt$$

and hence

$$\ln \left[ \frac{u_1(t)}{u_1(0)} \right] = u_2(t) - u_2(0).$$

Eventually, we obtain

$$u_1(t) = u_1(0)e^{u_2(t)-u_2(0)}. \quad (2.14)$$

Choose  $u_1(0) > 0$ . To prove that  $u_1(t) \neq 0$  for all  $t$  it is enough to show that  $u_2(t)$  is well defined (and in particular finite) for every  $t > 0$ . In the following we also prove that  $u_2(t)$  is also  $\mathcal{C}^\infty$  for every  $t \geq 0$ . This guarantees that no loss of regularity occurs.

Plugging (2.14) into the second line of (2.13) we get

$$du_2/dt = -\frac{u_2}{u_1(0)}e^{u_2(0)-u_2(t)}. \quad (2.15)$$

Note that  $u_2 = 0$  is an equilibrium for (2.15). Also, if  $u_2(0) < 0$  then  $du_2/dt \geq 0$  and hence  $u_2(0) \leq u_2(t) < 0$  for every  $t$ . Conversely, if  $u_2(0) > 0$  then  $du_2/dt \leq 0$  and hence  $0 \leq u_2(t) < u_2(0)$  for every  $t$ . In both cases, we get that  $u_2(t)$  is well defined and regular for every  $t \geq 0$ .

### 2.2.3 Example (2.16)

With Example (2.2.3) we discuss a system which satisfies Hypotheses 1 ... 6, but does not satisfy Hypothesis 7. As in Example (2.11), we exhibit a solution of this system which is not continuously differentiable and the loss of regularity regards a solution  $U$  such that  $\zeta[U(0)] \neq 0$ , but  $\zeta(U)$  reaches the value 0 for a finite value of  $t$ .

Consider the following system:

$$\begin{cases} du_1/dt = -u_3 \\ du_2/dt = -u_2/u_1 \\ du_3/dt = -u_3 \end{cases} \quad (2.16)$$

Let  $U = (u_1, u_2, u_3)^T$ ,  $\zeta(U) = u_1$  and

$$\phi_s(U) = \begin{pmatrix} 0 \\ -u_2 \\ 0 \end{pmatrix} \quad \phi_{ns}(U) = \begin{pmatrix} -u_3 \\ 0 \\ -u_3 \end{pmatrix}.$$

System (2.11) can then be written in the form

$$\frac{dU}{dt} = \frac{1}{\zeta(U)}\phi_s(U) + \phi_{ns}(U)$$

if the function  $F(U)$  defined by (2.2) is

$$F(U) = \begin{pmatrix} -u_3 u_1 \\ -u_2 \\ -u_3 u_1 \end{pmatrix}.$$

By direct check, one can verify that Hypotheses 1 ... 6 are verified by (2.16). On the other side, Hypothesis 7 is not satisfied in this case. Indeed, the surface  $\mathcal{S} = \{U : \zeta(U) = 0\}$  is the plane  $\{u_1 = 0\}$ . Thus, the set of points such that  $\zeta(U) = 0$  and  $F(U) = \vec{0}$  is  $\{u_1 = u_2 = 0\}$  and

$$\nabla \zeta \cdot DF \cdot (\nabla \zeta)^T = -u_3$$

is in general different from zero on this line.

An explicit solution of (2.16) can be obtained as follows. From the third and the first equation we get respectively

$$\begin{aligned} u_3(t) &= u_3(0)e^{-t} \\ u_1(t) &= u_1(0) - u_3(0) + u_3(0)e^{-t}. \end{aligned}$$

Assume that  $u_3(0) = Au_1(0)$  for some constant  $A$  whose exact value is determined in the following. The equation satisfied by  $u_2$  becomes

$$du_2/dt = -\frac{u_2}{Au_1(0)e^{-t} + u_1(0)(1-A)}.$$

Thus, we obtain

$$\frac{d}{dt} \left[ \ln(u_2(t)) \right] = \frac{1}{u_1(0)(A-1)} \frac{d}{dt} \left[ \ln(u_1(0)(1-A)e^t + Au_1(0)) \right]$$

and hence

$$u_2(t) = u_2(0) \left[ (1-A)e^t + A \right]^{1/(A-1)u_1(0)}.$$

If  $(A-1)u_1(0) > 1$ , then the first derivative  $du_2/dt$  blows up at  $t = \ln(A/A-1)$ . Note that this is exactly the value of  $t$  at which  $u_1(t)$  attains 0.

### 2.3 The case of the compressible Navier Stokes in Eulerian coordinates

In this section we show that Hypotheses 1 ... 7 are satisfied by the viscous profiles of the compressible Navier Stokes equation written in Eulerian coordinates. Note that the case of the Navier Stokes written in Lagrangian coordinates was already discussed for example in [13]. When the equation is formulated using Lagrangian coordinates, the ODE satisfied by the viscous profiles is not singular and hence one does not need the machinery developed in this work.

The compressible Navier Stokes written in Eulerian coordinates is (see e.g. [13]):

$$\begin{cases} \rho_t - v_x = 0 \\ (\rho v)_t + (\rho v^2 + p)_x = (\nu v_x)_x \\ \left( \rho e + \rho \frac{v^2}{2} \right)_t + \left( v \left[ \frac{1}{2} \rho v^2 + \rho e + p \right] \right)_x = (k \theta_x + \nu v v_x)_x \end{cases} \quad (2.17)$$

The unknowns are  $\rho(t, x)$ ,  $v(t, x)$  and  $e(t, x)$ . The function  $\rho$  is the density of the fluid,  $v$  represents the speed of the particles in the fluid and  $e$  is the internal energy. The function  $p = p(\rho, e) > 0$  is the pressure and satisfies  $p_\rho > 0$ , while  $\theta$  is the absolute temperature and in the case of a polytropic gas satisfies

$$\theta = \frac{e(\gamma-1)}{R},$$

where  $R$  is the universal gas constant and  $\gamma$  is a constant specific of the gas. Finally,  $\nu(\rho) > 0$  and  $k(\rho) > 0$  are respectively the viscosity and the heat conduction coefficient.

Equation (2.17) takes the form

$$E(u)u_t + A(u, u_x)u_x = B(u)u_{xx}, \quad (2.18)$$

for suitable matrices  $E$ ,  $A$  and  $B$ . For the applications described in [5], it is convenient to have that  $A(U, \vec{0})$  is *symmetric*. We thus multiply (2.17) on the left for a suitable positive definite matrix and we eventually obtain that it can be written in the form (2.18) for

$$E(\rho, v, e) = \begin{pmatrix} p_\rho/\rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_e/\rho \end{pmatrix} \quad B(\rho, v, e) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu/\rho & 0 \\ 0 & 0 & (\gamma-1)k/R\rho \end{pmatrix} \quad (2.19)$$

and

$$A(\rho, v, e, \rho_x, v_x, e_x) = \begin{pmatrix} p_\rho v/\rho^2 & p_\rho/\rho & 0 \\ p_\rho/\rho & v - \nu'\rho_x/\rho & p_e/\rho \\ 0 & p_e/\rho - \nu v_x p_e/(\rho p) & \nu p_e/\rho - (\gamma-1)k'p_e\rho_x/(R\rho p) \end{pmatrix} \quad (2.20)$$

In the previous expression, we denote by  $p_\rho$  and  $p_e$  the partial derivative of  $p$  with respect to  $\rho$  and  $e$  respectively. In the following, to simplify the notations we write

$$A_{21} = \begin{pmatrix} p_\rho/\rho \\ 0 \end{pmatrix} \quad A_{22} = \begin{pmatrix} v - \nu'\rho_x/\rho & p_e/\rho \\ p_e/\rho - \nu v_x p_e/(\rho p) & \nu p_e/\rho - (\gamma-1)k'p_e\rho_x/(R\rho p) \end{pmatrix} \quad (2.21)$$

and

$$b = \begin{pmatrix} \nu/\rho & 0 \\ 0 & (\gamma-1)k/R\rho \end{pmatrix} \quad a_{11} = p_\rho/\rho^2.$$

In the following we also suppose that  $\rho$  is bounded away from 0, say  $\rho \geq c_\rho > 0$  for a suitable constant  $c_\rho$ . This implies that vacuum states are not reached.

In the following we focus on the ordinary differential equation satisfied by the steady solutions

$$Au_x = Bu_{xx} \quad (2.22)$$

in the case  $A$  and  $B$  are given by (2.20) and (2.19) respectively. Another class of viscous profiles considered in [5] are travelling waves: the analysis in that case is very similar, just the notations are more complicated.

We set

$$w = \rho_x \quad \vec{z} = (v_x, e_x)^T \quad (2.23)$$

Then (2.22) becomes

$$\begin{pmatrix} A_{11}(u) & A_{21}^T(u) \\ A_{21}(u) & A_{22}(u, u_x) \end{pmatrix} \begin{pmatrix} w \\ \vec{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b(u) \end{pmatrix} \begin{pmatrix} w_x \\ \vec{z}_x \end{pmatrix},$$

i.e.

$$\begin{cases} a_{11}vw + A_{21}^T\vec{z} = 0 \\ A_{21}w + A_{22}\vec{z} = b\vec{z}_x \end{cases}$$

Assume  $v \neq 0$ , then (2.22) can be written as

$$\begin{cases} w = -\frac{A_{21}^T\vec{z}}{a_{11}v} \\ \vec{z}_x = b^{-1}\left[A_{22} - \frac{A_{21}A_{21}^T}{a_{11}v}\right]\vec{z} \end{cases} \quad (2.24)$$

Note that the matrix  $b$  is invertible and hence the previous expression is well defined.

Combining (2.23) and (2.24), one obtains that the equation satisfied by the viscous profiles of the compressible Navier Stokes can be written in the form

$$\frac{dU}{dx} = \frac{1}{\zeta(U)} F(U)$$

provided that  $U = (\rho, v, e, \vec{z})^T$ ,  $\zeta(U) = v$  and

$$F(U) = \begin{pmatrix} A_{21}^T \vec{z}/a_{11} \\ v \vec{z} \\ b^{-1} [A_{22}v - A_{21}A_{21}^T/a_{11}] \vec{z} \end{pmatrix}$$

Note that  $A_{22}$  depends on  $\rho_x$  but, plugging  $w = -A_{21}^T \vec{z}/(a_{11}v)$  into (2.21) one gets that  $A_{22}$  evaluated at a point  $(\rho, v = 0, e, \vec{z} = \vec{0})$  is the null matrix.

Thus, the jacobian  $DF$  satisfies

$$DF(\rho, v = 0, e, \vec{z} = \vec{0}) = \begin{pmatrix} 0 & -A_{21}^T/a_{11} \\ \vec{0} & \mathbf{0}_2 \\ \vec{0} & -b^{-1}A_{21}A_{21}^T/a_{11} \end{pmatrix},$$

where  $\mathbf{0}_2$  denotes the  $2 \times 2$  null matrix. Since  $A_{21}A_{21}^T/a_{11}$  admits only eigenvalues with strictly positive real part, then  $DF$  admits only eigenvalues with non positive real part and hence Hypothesis 1 is satisfied. Also, the dimension of every centre manifold of

$$\frac{dU}{d\tau} = F(U)$$

is 3. Since the subspace  $\{\vec{z} = \vec{0}\}$  is entirely constituted by equilibria for the equation, it coincides with the centre manifold. Thus, Hypothesis 4 is satisfied. Since  $\zeta(U) = v$ , then Hypothesis 3 is also verified. Concerning Hypothesis 5, this is satisfied because  $\{\vec{z} = \vec{0}\}$  is transversal to the singular surface  $\{v = 0\}$ . Finally, by direct check one can show that also Hypotheses 6 and 7 are verified. Thus the machinery developed in this work applies to the viscous profiles with small total variation of the compressible Navier Stokes equation written in Eulerian coordinates.

### 3 Change of coordinates

In this section we construct a change of coordinates which allows to write system

$$\frac{dU}{d\tau} = F(U) \tag{3.1}$$

in a more convenient form. Before stating our result, we introduce some notations: as before,  $N$  denotes the dimension of  $U$ . Also,  $n_-$  is the number of eigenvalues of  $DF(\vec{0})$  with strictly negative real part, while  $(n_0 + 1)$  is the number of eigenvalues of  $DF(\vec{0})$  with zero real part. Thanks to Hypothesis 1,  $N = n_- + n_0 + 1$

**Proposition 3.1.** *Let Hypotheses 1 ... 7 hold. Then there exists a strictly positive constant  $\delta$ ,  $\delta \ll 1$  such that the following holds. There exists smooth map*

$$\Upsilon : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

*which is invertible in a neighbourhood of  $\vec{0}$  of size  $\delta$ . Write  $\Upsilon(U) = \bar{U}$  as a column vector:*

$$\bar{U} = \begin{pmatrix} \bar{\zeta} \\ \bar{u}_0 \\ \bar{u}_- \end{pmatrix},$$

where  $\zeta \in \mathbb{R}$ ,  $u_0 \in \mathbb{R}^{n_0}$  and  $u_- \in \mathbb{R}^{n_-}$ . If  $U$  satisfies (3.1) then  $\bar{U}$  satisfies

$$\begin{cases} d\bar{\zeta}/d\tau = \bar{G}_{10}(\bar{\zeta}, \bar{u}_0)\bar{u}_0\bar{\zeta}^2 + G_{1-}(\bar{\zeta}, \bar{u}_0, \bar{u}_-)\bar{u}_-\bar{\zeta} \\ d\bar{u}_0/d\tau = \left\{ \bar{G}_{01}(\bar{\zeta}, \bar{u}_0) + [\bar{G}_{0-}(\bar{\zeta}, \bar{u}_0, \bar{u}_-) - \bar{G}_{0-}(\bar{\zeta}, \bar{u}_0, \vec{0})] \right\} \bar{\zeta}\bar{u}_0 \\ d\bar{u}_-/d\tau = G_s(\bar{\zeta}, \bar{u}_0, \bar{u}_-)\bar{u}_- \end{cases} \quad (3.2)$$

In the previous expression,  $G_{10}$  is a row vector belonging to  $\mathbb{R}^{n_0}$ ,  $G_{1-}$  is a row vector in  $\mathbb{R}^{n_-}$ , the matrices  $G_{01}$  and  $G_{0-}$  belong to  $\mathbb{M}^{n_0 \times n_0}$  and the matrix  $G_s$  belongs to  $\mathbb{M}^{n_- \times n_-}$ . Also, all the eigenvalues of the matrix  $G_s(0, \vec{0}, \vec{0})$  have strictly negative real part.

### 3.0.1 Notations and preliminary results

Before proving Proposition 3.1, we introduce two preliminary results (Lemmata 3.1 and 3.2) and we go over the definition of *uniformly stable manifold*.

Let  $B_\delta(\vec{0}) \subset \mathbb{R}^N$  denote the open ball with centre  $\vec{0}$  and radius  $\delta$ . Let

$$\tilde{\Upsilon} : B_\delta(\vec{0}) \rightarrow \mathbb{R}^N$$

a smooth map which is invertible on the set of images. To simplify the exposition, we also assume that  $\tilde{\Upsilon}(\vec{0}) = \vec{0}$ . Let  $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$  be a smooth cut-off function such that

$$\psi(U) = \begin{cases} 1 & |U| \leq \delta/2 \\ 0 & |U| \geq \delta. \end{cases}$$

In the previous expression,  $|U|$  denotes the modulus of  $U$ . Let

$$\tilde{\Upsilon}(U) := \psi(|U|)\tilde{\Upsilon}(U) + [1 - \psi(|U|)]D\tilde{\Upsilon}(0)U, \quad (3.3)$$

where  $D\tilde{\Upsilon}(0)$  is the jacobian computed at the point  $\vec{0}$ . Note that  $\tilde{\Upsilon}$  coincides with  $\tilde{\Upsilon}$  on  $B_{\delta/2}(\vec{0})$ . Also,

$$\|D\tilde{\Upsilon}\|_{C^0} \leq C. \quad (3.4)$$

for a suitable constant  $C$ . Let  $\bar{U} := \tilde{\Upsilon}(U)$  and

$$\bar{F}(\bar{U}) := D\tilde{\Upsilon}[\tilde{\Upsilon}^{-1}(\bar{U})]F[\tilde{\Upsilon}^{-1}(\bar{U})] \quad (3.5)$$

If the function  $U(\tau)$  satisfies (3.1), then  $\bar{U}(\tau)$  solves

$$\frac{d\bar{U}}{d\tau} = \bar{F}(\bar{U}). \quad (3.6)$$

Also, given a real valued function  $\zeta(\bar{U})$ , let

$$\bar{\zeta}(\bar{U}) := \zeta[\tilde{\Upsilon}^{-1}(\bar{U})]. \quad (3.7)$$

**Lemma 3.1.** *Assume that Hypotheses 1 ... 7 are satisfied by  $F$  and  $\zeta$ . Then the same hypotheses are verified by  $\bar{F}$  and  $\bar{\zeta}$ .*

*Proof.* Since

$$D\bar{F}(\vec{0}) = D\tilde{\Upsilon}(\vec{0})DF(\vec{0})[D\tilde{\Upsilon}(\vec{0})]^{-1},$$

then the eigenvalues of  $D\bar{F}(\vec{0})$  are the same as the eigenvalues of  $DF(\vec{0})$  and hence Hypothesis 1 is satisfied by  $\bar{F}$ .

Let  $\mathcal{M}^c$  be a centre manifold for (3.1) around the equilibrium point  $\vec{0}$ . Exploiting (3.4) one can then show that  $\tilde{\Upsilon}(\mathcal{M}^c)$  is a centre manifold for (3.6). Also,

$$\bar{\mathcal{S}} := \{\bar{U} : \bar{\zeta}(\bar{U}) = 0\} = \bar{\Upsilon}(\mathcal{S}), \quad (3.8)$$

where  $\mathcal{S}$  is the set defined by (2.6). Since  $\tilde{\Upsilon}$  is one to one,

$$\tilde{\Upsilon}(\mathcal{S}) \cap \tilde{\Upsilon}(\mathcal{M}^c) = \tilde{\Upsilon}(\mathcal{S} \cap \mathcal{M}^c).$$

Since all the points  $U$  belonging to  $\mathcal{S} \cap \mathcal{M}^c$  satisfy  $F(U) = \vec{0}$ , from (3.5) and from Hypothesis 4 it follows that all the points  $\bar{U}$  belonging to  $\tilde{\Upsilon}(\mathcal{S}) \cap \tilde{\Upsilon}(\mathcal{M}^c)$  satisfy  $\bar{F}(\bar{U}) = \vec{0}$ . Thus, Hypothesis 4 is satisfied by  $\bar{F}$  and  $\bar{\zeta}$ .

Let us focus on Hypothesis 5. Let  $\mathcal{M}^{eq}$  be a manifold of equilibria for (3.1) which is transversal to the manifold  $\mathcal{S}$  defined by (2.6). Then  $\tilde{\Upsilon}(\mathcal{M}^{eq})$  is a manifold of equilibria for (3.6). Also, let  $\bar{\mathcal{S}}$  as in (3.8), then  $\tilde{\Upsilon}(\mathcal{M}^{eq})$  is transversal to  $\bar{\mathcal{S}}$ . Indeed,

$$\tilde{\Upsilon}(\mathcal{M}^{eq}) \cap \bar{\mathcal{S}} = \tilde{\Upsilon}(\mathcal{M}^{eq} \cap \mathcal{S})$$

is a manifold with dimension  $N - 1$ .

By direct check, one verify that also Hypotheses 3, 6 and 7 are verified by  $\bar{F}$  and  $\bar{\zeta}$ .  $\square$

To prove the regularity of the change of variable described in the statement of Proposition 3.1 we need the following result. Let  $X$  be a closed subset of a Banach space  $\tilde{X}$  and let  $Y$  be another Banach space. Also, let

$$T : X \times Y \rightarrow \tilde{X}$$

be a map such that, for every  $y \in Y$ ,  $T(\cdot, y)$  takes values in  $X$  and is a contraction. Thanks to the contraction map theorem, we can define a map

$$Y \rightarrow X$$

which associates to  $y$  the fixed point of the map  $T(\cdot, y)$ . We denote this map by  $x(y)$  and we are interested in its regularity. Assume, that, for every  $y$ ,  $x(y)$  belongs to the inner part of  $X$ . Also, assume that, for every  $(\bar{x}, \bar{y}) \in X \times Y$  such that  $\bar{x}$  is an inner point,  $T(\cdot, \bar{y})$  is Fréchet differentiable at  $\bar{x}$  and denote by  $T_x(\bar{x}, \bar{y}) \in \mathcal{L}(\tilde{X}, \tilde{X})$  its derivative. Also, assume that, for every  $(\bar{x}, \bar{y})$  in the same conditions as before, the map  $T(\bar{x}, \cdot)$  is Fréchet differentiable at  $\bar{y}$  and denote by  $T_y(\bar{x}, \bar{y}) \in \mathcal{L}(Y, \tilde{X})$  its derivative.

**Lemma 3.2.** *Assume that the maps  $T_x(\bar{x}, \bar{y})$  and  $T_y(\bar{x}, \bar{y})$  are both continuous as maps from the inner part of  $X \times Y$  into respectively  $\mathcal{L}(\tilde{X}, \tilde{X})$  and  $\mathcal{L}(Y, \tilde{X})$ . Then the map  $x(y)$  is continuously differentiable and the Fréchet derivative at  $\bar{y}$  is*

$$\left( I - T_x[(x(\bar{y}), \bar{y})] \right)^{-1} \circ T_y[(x(\bar{y}), \bar{y})], \quad (3.9)$$

where  $I$  denotes the identity.

The proof is given for completeness.

*Proof.* Fix  $\bar{y} \in Y$  and  $h \in Y$ . Then

$$\begin{aligned} x(\bar{y} + h) - x(\bar{y}) &= T(x(\bar{y} + h), \bar{y} + h) - T(x(\bar{y}), \bar{y}) \\ &= T(x(\bar{y} + h), \bar{y} + h) - T(x(\bar{y} + h), \bar{y}) + T(x(\bar{y} + h), \bar{y}) - T(x(\bar{y}), \bar{y}) \\ &= T_y(x(\bar{y} + h), \bar{y})[h] + T_x(x(\bar{y}), \bar{y})[x(\bar{y} + h) - x(\bar{y})] + o(\|h\|_Y) \\ &= T_y(x(\bar{y}), \bar{y})[h] + T_x(x(\bar{y}), \bar{y})[x(\bar{y} + h) - x(\bar{y})] + o(\|h\|_Y). \end{aligned}$$

Thus

$$\left( I - T_x[(x(\bar{y}), \bar{y})] \right) [x(\bar{y} + h) - x(\bar{y})] = T_y[(x(\bar{y}), \bar{y})] + o(\|h\|_Y).$$

To conclude, it is enough to observe that, being  $T(\cdot, \bar{y})$  a contraction, then  $\|T_x(x(\bar{y}), \bar{y})\|_{\mathcal{L}(X, X)} < 1$ . Thus, thanks for example to Proposition 1.1 page 31 in [1], the map  $I - T_x[(x(\bar{y}), y)]$  is invertible. The map  $x(y)$  is then Fréchet differentiable at  $\bar{y}$ , with derivative given by (3.9). The continuity of the Fréchet derivative as a map taking values in  $\mathcal{L}(Y, X)$  is a consequence of the continuity of  $T_x$  and  $T_y$ .  $\square$

In the proof of Proposition 3.1 we also exploit the existence of the so called *uniformly stable manifold*. The existence of such a manifold is implied by Hadamard-Perron theorem, which is discussed for example in [9]. The use of the uniformly stable manifold to study the viscous profiles is derived from [2]. Here we just recall some of the most important properties. Let  $E$  be a manifold of equilibria for the ordinary differential equation

$$\frac{dU}{d\tau} = F(U) \quad U \in \mathbb{R}^N.$$

Assume that  $F(\vec{0}) = \vec{0}$  and that  $\vec{0}$  belongs to  $E$ . Denote by  $V^s$  the eigenspace corresponding to eigenvalues of  $DF(\vec{0})$  with strictly negative real part. Then in a sufficiently small neighbourhood of  $\vec{0}$  one can define a manifold, called uniformly stable manifold, which is parameterized by  $V^s \oplus E$  and satisfies the following. If  $U_0$  belongs to this manifold, denote by  $U(\tau)$  the solution of the Cauchy problem

$$\begin{cases} dU/d\tau = F(U) \\ U(0) = U_0 \end{cases}$$

Then there exists  $\bar{U}$  belonging to  $E$  such that

$$\lim_{\tau \rightarrow +\infty} e^{c/2\tau} |U(\tau) - \bar{U}| = 0.$$

In the previous expression,  $c > 0$  is a constant such that  $\operatorname{Re} \lambda < -c$  for every  $\lambda$  eigenvalue of  $DF(\vec{0})$  with strictly negative real part. Also, the uniformly stable manifold is tangent to  $V^s \oplus E$  at  $\vec{0}$ .

### 3.0.2 Proof of Proposition 3.1: first part

We are now ready to prove Proposition 3.1. The proof is actually a bit long, so we divide it into two parts. In this section we discuss the first part, the second part is given in Section 3.0.3. We proceed in several steps.

- *Step one:* thanks to Hypothesis 3,  $\nabla \zeta(\vec{0}) \neq \vec{0}$ . Let  $U = (u_1 \dots u_N)^T$  be the components of  $U$ . Just to fix the ideas, we can assume

$$\frac{\partial \zeta}{\partial u_1}(\vec{0}) \neq 0. \tag{3.10}$$

Define

$$\tilde{\Upsilon}_1(U) = \begin{pmatrix} \zeta(U) \\ u_2 \\ \dots \\ u_N \end{pmatrix}.$$

Thanks to (3.10),  $D\tilde{\Upsilon}_1(\vec{0})$  is invertible and hence  $\tilde{\Upsilon}_1$  define a change of coordinates in a neighbourhood of  $\vec{0}$ , say in  $B_\delta(\vec{0})$  provided that  $\delta$  is small enough. We can then introduce the functions  $\tilde{\Upsilon}$ ,  $\bar{F}$  and  $\bar{\zeta}$  defined by (3.3), (3.5) and (3.7) respectively. Note that in this case

$$\bar{\zeta}(\bar{U}) = \bar{u}_1,$$

where  $\bar{u}_1$  is the first component of  $\bar{U}$ . Thanks to Lemma 3.1, Hypotheses 1 ... 7 are satisfied by the ordinary differential equation

$$\frac{d\bar{U}}{d\tau} = \bar{F}(\bar{U}).$$

In the following, to simplify the notations we write  $U$  and  $F$  instead  $\bar{U}$  and  $\bar{F}$ .

- *Step 2:* thanks to Hypothesis 5, there exists a manifold  $\mathcal{M}^{eq}$  which is entirely constituted by equilibria and which is transversal to the manifold  $\mathcal{S}$ , namely to  $\{u_1 = 0\}$ . Let  $n_{eq}$  be the dimension of this manifold. Also, let

$$\phi^{eq} : \mathbb{R}^N \rightarrow \mathbb{R}^{N-n_{eq}}$$

a function such that  $U \in \mathcal{M}^{eq}$  if and only  $\phi^{eq}(U) = \vec{0}$ . Since  $\vec{0}$  belongs to  $\mathcal{M}^{eq}$ , then  $\phi^{eq}(\vec{0}) = \vec{0}$ . Let

$$\phi : \mathbb{R}^N \rightarrow \mathbb{R}^{N+1-n_{eq}}$$

be defined by

$$\phi(U) = \begin{pmatrix} \zeta \\ \phi^{eq}(U) \end{pmatrix}.$$

From the fact that  $\mathcal{M}^{eq}$  and the manifold  $\{\zeta = 0\}$  are transversal we deduce that  $D\phi$  has rank  $N - n_{eq} + 1$  when both  $\zeta = 0$  and  $\phi^{eq} = \vec{0}$ , thus in particular when  $U = \vec{0}$ . To fix the ideas, assume that the first and the last  $N - n_{eq}$  columns of  $D\phi(\vec{0})$  are linearly independent. Let

$$\tilde{\Upsilon}^2(U) := \begin{pmatrix} \zeta \\ u_2 \\ \dots \\ u_{n_{eq}} \\ \phi^{eq}(U) \end{pmatrix},$$

then because of the previous considerations the jacobian  $D\tilde{\Upsilon}^2(\vec{0})$  is invertible and hence we can define  $\bar{U}$  and  $\bar{F}$  in the same way as in Step one. Note that as before  $\bar{\zeta}(\bar{U}) = u_1$ . Also, the manifold

$$E := \{\bar{u}_2 = \dots = \bar{u}_N = 0\} \tag{3.11}$$

is now entirely constituted by equilibria. In the following, to simplify the notations we write  $U$  and  $F$  instead  $\bar{U}$  and  $\bar{F}$ . Hypotheses 1 ... 7 are satisfied because of Lemma 3.1.

- *Step Three:* let  $E$  be as in (3.11) and denote by  $V^c$  the eigenspace of  $DF(\vec{0})$  associated to eigenvalues with 0 real part. Also, let  $V^s$  is the eigenspace associated to eigenvalues with strictly negative real part. The dimension of  $V^c$  and of  $V^s$  is respectively  $n_0 + 1$  and  $n_-$ . Thanks to Hypothesis 1,  $N = n_0 + 1 + n_-$ . The vector  $(1, 0 \dots 0)$  belongs to  $V^c$  because  $E \subseteq V^c$ . Also, we can assume, *via* a linear change of coordinates, that

$$V^c = \{u_{N+1-n_-} = \dots u_N = 0\} \quad V^s = \{\zeta = 0, u_2 = \dots u_{n_0+1} = 0\}.$$

Fix a centre manifold  $\mathcal{M}^c$  for the system

$$\frac{dU}{d\tau} = F(U) \tag{3.12}$$

around the equilibrium point  $\vec{0}$ . Let

$$\phi^c : V^c \rightarrow \mathbb{R}^N$$

be a map that parameterizes  $\mathcal{M}^c$ : it can be chosen in such a way that the following holds. Denote by  $\phi_1^c \dots \phi_N^c$  the components of  $\phi^c$ . Then  $\phi_1^c(\zeta, u_2 \dots u_{1+n_0}) = u_1, \dots \phi_{1+n_0}^c(\zeta, u_2 \dots u_{1+n_0}) = u_{1+n_0}$ . In other words, let  $\phi^0 : \mathbb{R}^N \rightarrow \mathbb{R}^{n_-}$  the map defined by

$$\phi^0(\zeta, u_2, \dots u_N) = \begin{pmatrix} u_{N+1-n_-} - \phi_{N+1-n_-}^c \\ \dots \\ u_N - \phi_N^c \end{pmatrix}.$$



Then  $U$  belongs to  $\mathcal{M}^c$  if and only if  $\phi^0(U) = \vec{0}$ . Also, since  $\mathcal{M}^c$  is tangent at  $\vec{0}$  to  $V^c$ , then the jacobian  $D\phi^0(\vec{0})$  satisfies

$$D\phi^0(\vec{0}) = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

Let  $\mathcal{M}_E^{us}$  be the uniformly stable manifold of system (3.12) around the equilibrium point  $\vec{0}$ . The manifold  $\mathcal{M}_E^{us}$  is parameterized by  $E \oplus V^s$  and it is tangent to this space at  $\vec{0}$ . Proceeding as before, we define a map

$$\phi_E^{us} : \mathbb{R}^N \rightarrow \mathbb{R}^{n_0}$$

such that  $U \in \mathcal{M}_E^{us}$  if and only if  $\phi_E^{us}(U) = \vec{0}$ . Also, if we set

$$\tilde{\Upsilon}^3(U) := \begin{pmatrix} \zeta \\ \phi_E^{us}(U) \\ \phi^0(U) \end{pmatrix},$$

then the jacobian  $D\tilde{\Upsilon}(\vec{0})$  is the identity. We can thus proceed as in Step 2 and we can define  $\bar{U}$  and  $\bar{F}(\bar{U})$ . In the following, to simplify the notations we write  $U$  and  $F$  instead of  $\bar{U}$  and  $\bar{F}$ . Note that the Hypotheses 1 ... 7 are satisfied because of Lemma 3.1.

- *Step four:* consider the following decomposition:

$$U = \begin{pmatrix} \zeta \\ u_0 \\ u_- \end{pmatrix} \quad F(U) = \begin{pmatrix} f_1(\zeta, u_0, u_-) \\ F_0(\zeta, u_0, u_-) \\ F_-(\zeta, u_0, u_-) \end{pmatrix} \quad (3.13)$$

where  $\zeta, f_1 \in \mathbb{R}$ ,  $u_0, F_0 \in \mathbb{R}^{n_0}$  and  $u_-, F_- \in \mathbb{R}^{n_-}$ . In the new coordinates, the centre manifold  $\mathcal{M}^c$  is the subspace  $\{u_- = \vec{0}\}$  and the uniformly stable manifold  $\mathcal{M}_E^{us}$  is  $\{u_0 = \vec{0}\}$ .

The centre manifold is invariant for the equation

$$\frac{dU}{d\tau} = F(U) \quad (3.14)$$

and hence  $F_-(\zeta, u_0, \vec{0}) = \vec{0}$  for every  $\zeta$  and  $u_0$ . By regularity,

$$F_-(\zeta, u_0, u_-) = G_s(\zeta, u_0, u_-)u_-$$

for a suitable matrix  $G_s \in \mathbb{M}^{n_- \times n_-}$ . Also, the uniformly stable manifold is invariant and hence proceeding as before we get that

$$F_0(\zeta, u_0, u_-) = G_c(\zeta, u_0, u_-)u_0$$

for a suitable matrix  $G_c \in \mathbb{M}^{n_0 \times n_0}$ . Finally, Hypothesis 6 implies that

$$f_1(0, u_0, u_-) = 0$$

and hence by regularity  $f_1(\zeta, u_0, u_-) = g_1(\zeta, u_0, u_-)\zeta$ . Consider the decomposition

$$G_c(\zeta, u_0, u_-) = G_c(\zeta, u_0, \vec{0}) + [G_c(\zeta, u_0, u_-) - G_c(\zeta, u_0, \vec{0})].$$

Thanks to Hypothesis 4, the subspace  $\{\zeta = 0, u_- = \vec{0}\}$  is entirely constituted by equilibria and hence

$$G_c(0, u_0, \vec{0}) = \vec{0}.$$

By regularity,  $G_c(\zeta, u_0, \vec{0}) = G_{01}(\zeta, u_0)\zeta$  for a suitable matrix  $G_{01} \in \mathbb{M}^{n_0 \times n_0}$ . Putting all the previous considerations together, we get that system (3.14) can be written as

$$\begin{cases} d\zeta/d\tau = g_1(\zeta, u_0, u_-)\zeta \\ du_0/d\tau = \left\{ G_{01}(\zeta, u_0)\zeta + [G_c(\zeta, u_0, u_-) - G_c(\zeta, u_0, \vec{0})] \right\} u_0 \\ du_-/d\tau = G_s(\zeta, u_0, u_-)u_- \end{cases} \quad (3.15)$$

Consider the decomposition

$$g_1(\zeta, u_0, u_-) = g_1(\zeta, u_0, \vec{0}) + [g_1(\zeta, u_0, u_-) - g_1(\zeta, u_0, \vec{0})]$$

By construction  $G_s(\vec{0})$  admits only eigenvalues with strictly negative real part, thus  $G_s(\zeta, u_0, u_-)u_- = \vec{0}$  implies  $u_- = \vec{0}$ . Thus, the set  $\{U : \zeta(U) = 0, F(U) = \vec{0}\}$  is the subspace  $\{\zeta = 0, u_- = \vec{0}\}$ . Thanks to Hypothesis 7, we have

$$g_1(0, u_0, \vec{0}) = 0.$$

By regularity, we thus have

$$g_1(\zeta, u_0, \vec{0}) = g_{11}(\zeta, u_0)\zeta \quad [g_1(\zeta, u_0, u_-) - g_1(\zeta, u_0, \vec{0})] = G_{1-}(\zeta, u_0, u_-)u_-$$

for a suitable row vector  $G_{1-}(\zeta, u_0, u_-) \in \mathbb{R}^{n_-}$ . Also, since the manifold  $\{u_0 = \vec{0}, u_- = \vec{0}\}$  is entirely constituted by equilibria, then  $g_{11}(\zeta, \vec{0}) = 0$  for every  $\zeta$  and hence

$$g_{11}(\zeta, u_0) = G_{10}(\zeta, u_0)u_0$$

for a suitable vector  $G_{10} \in \mathbb{R}^{n_0}$ . In other words, (3.15) reduces to

$$\begin{cases} d\zeta/d\tau = \zeta^2 G_{10}(\zeta, u_0)u_0 + \zeta G_{1-}(\zeta, u_0, u_-)u_- \\ du_0/d\tau = \left\{ G_{01}(\zeta, u_0)\zeta + [G_c(\zeta, u_0, u_-) - G_c(\zeta, u_0, \vec{0})] \right\} u_0 \\ du_-/d\tau = G_s(\zeta, u_0, u_-)u_- \end{cases} \quad (3.16)$$

### 3.0.3 Proof of Proposition 3.1: second part

In this section we complete the proof of Proposition 3.1 introducing a refined change of coordinates.

Consider system (3.15) restricted on the invariant subspace  $\{\zeta = 0\}$ . One obtains

$$\begin{cases} du_0/d\tau = [G_c(0, u_0, u_-) - G_c(0, u_0, \vec{0})] u_0 \\ du_-/d\tau = G_s(0, u_0, u_-)u_- \end{cases} \quad (3.17)$$

The subspace  $\{u_- = \vec{0}\}$  is entirely constituted by equilibria. Also, given a point  $(u_0, u_-)$  belonging to a small enough neighbourhood of  $\vec{0}$ , then the solution of (3.17) starting at  $(u_0, u_-)$  decays exponentially fast to a point in the subspace  $\{u_- = \vec{0}\}$ . This is a consequence of the fact that  $G_s(0, \vec{0}, \vec{0})$  admits only eigenvalues with strictly negative real part.

We want to define a change of coordinates  $\bar{U} = \tilde{\Upsilon}^4(U)$  such that in the new coordinates  $\bar{U}$  the following holds. For every  $\bar{u}_0(0) \in \mathbb{R}^{n_0}$  and for every  $\bar{u}_-(0) \in \mathbb{R}^{n_-}$ , the solution of (3.17) starting at the point  $(\bar{u}_0(0), \bar{u}_-(0))$  converges exponentially fast to the point  $(\bar{u}_0(0), \vec{0})$ . In other words, the set

$\{\bar{u}_0 = \bar{u}_0(0)\}$  is the stable manifold of system (3.17) around the equilibrium point  $(\bar{u}_0(0), \vec{0})$ . Assume, for the time being, that such a change of coordinates  $\tilde{\Upsilon}^4$  exists and let

$$F(\bar{U}) = \begin{pmatrix} \bar{\zeta}^2 \bar{G}_{10}(\bar{\zeta}, \bar{u}_0) \bar{u}_0 + \bar{\zeta} \bar{G}_{1-}(\bar{\zeta}, \bar{u}_0, \bar{u}_-) \bar{u}_- \\ \left\{ \bar{G}_{01}(\bar{\zeta}, \bar{u}_0) \bar{\zeta} + [\bar{G}_c(\bar{\zeta}, \bar{u}_0, \bar{u}_-) - \bar{G}_c(\bar{\zeta}, \bar{u}_0, \vec{0})] \right\} \bar{u}_0 \\ G_s(\bar{\zeta}, \bar{u}_0, \bar{u}_-) \bar{u}_- \end{pmatrix}$$

Because of the previous considerations, when  $\bar{\zeta} = 0$  then  $d\bar{u}_0/d\tau = 0$  and hence

$$[\bar{G}_c(0, \bar{u}_0, \bar{u}_-) - \bar{G}_c(0, \bar{u}_0, \vec{0})] \bar{u}_0 = \vec{0}.$$

By regularity,

$$[\bar{G}_c(\bar{\zeta}, \bar{u}_0, \bar{u}_-) - \bar{G}_c(\bar{\zeta}, \bar{u}_0, \vec{0})] = [\bar{G}_{0-}(\bar{\zeta}, \bar{u}_0, \bar{u}_-) - \bar{G}_{0-}(\bar{\zeta}, \bar{u}_0, \vec{0})] \zeta$$

for a suitable function  $G_{0-} \in \mathbb{M}^{n_0 \times n_0}$ . This concludes the proof of Proposition 3.1: it is enough to define the map  $\tilde{\Upsilon}$  as  $\tilde{\Upsilon}^4 \circ \tilde{\Upsilon}^3 \circ \tilde{\Upsilon}^2 \circ \tilde{\Upsilon}^1$  and then use the extension (3.3).

To define the change of coordinates  $\tilde{\Upsilon}^4$  we proceed as follows.

Fix  $\underline{u}_0 \in \mathbb{R}^{n_0}$  and  $\underline{u}_- \in \mathbb{R}^{n_-}$ . The solution of (3.17) such that  $u_-(0) = \underline{u}_-$  and

$$\lim_{\tau \rightarrow +\infty} u_0(\tau) = \underline{u}_0$$

is the fixed point of the application  $T[u_0, u_-] = (T_1[u_0, u_-], T_2[u_0, u_-])$  defined as follows:

$$\begin{cases} T_1[u_0, u_-](\tau) = \underline{u}_0 - \int_{+\infty}^{\tau} [G_c(0, u_0(s), u_-(s)) - G_c(0, u_0(s), \vec{0})] u_0(s) ds \\ T_2[u_0, u_-](\tau) = e^{\bar{G}_s \tau} \underline{u}_- + \int_0^{\tau} e^{\bar{G}_s(\tau-s)} [\bar{G}_s - G_s(0, \bar{u}_0(s), \bar{u}_-(s))] u_-(s) ds \end{cases} \quad (3.18)$$

In the previous expression,  $\bar{G}_s = G_s(0, \vec{0}, \vec{0})$ . To prove that  $T$  admits indeed a fixed point one can exploit the contraction map theorem. Namely, fix a positive constant  $\delta < 1$  such that  $|\underline{u}_-| \leq \delta$  and  $|\underline{u}_0| \leq \delta$ . Let  $X_0$  denote the subset of functions  $u_0$  belonging  $C^0([0, +\infty[, \mathbb{R}^{n_0})$  that decay exponentially fast to some limit: for every  $u_0 \in X_0$  there exists a value  $\tilde{u}_0$  such that

$$\lim_{\tau \rightarrow +\infty} |u_0 - \tilde{u}_0| e^{c\tau/2} = 0.$$

The positive constant  $c > 0$  satisfy  $Re\lambda \leq -c$  for every  $\lambda$  eigenvalue of  $\bar{G}_s$ . We also impose that, if  $u_0 \in X_0$ , then  $\|u_0\|_0 \leq c_0 \delta$ , where the norm  $\|\cdot\|_0$  on  $X_0$  is defined as follows:

$$\|u_0\|_0 = |\tilde{u}_0| + \sup_{\tau} \{e^{c\tau/2} |u_0(\tau) - \tilde{u}_0|\}.$$

In the previous expression,  $\tilde{u}_0$  stands for the limit of  $u_0$ . We explain in the following how the exact value of the constant  $c_0$  has to be determined. Also, let

$$X_- = \left\{ u_- \in C^0([0, +\infty[, \mathbb{R}^{n_-}) : \|u_-\|_- \leq c_- \delta \right\},$$

where the norm  $\|\cdot\|_-$  on  $X_-$  is defined by

$$\|u_-\|_- = \sup_{\tau} \{e^{c\tau/2} |u_-(\tau)|\}.$$

If  $\delta$  is sufficiently small, then one can define the constants  $c_-$  and  $c_0$  in such a way that the following holds. If  $(u_0, u_-) \in X_0 \times X_-$ , then  $T(u_0, u_-) \in X_0 \times X_-$ . Also,  $T$  is a contraction with respect to

the norm  $\|(u_0, u_-)\| = \|u_0\|_0 + \|u_-\|_-$  with Lipschitz constant is less or equal to 1/2. The proof relies on standard computations. Thanks to the contraction map theorem,  $T$  admits a fixed point and hence we can define a map

$$\phi_- : \mathbb{R}^{n_0} \times \mathbb{R}^{n_-} \rightarrow \mathbb{R}^{n_0} \quad (3.19)$$

which associates to the couple  $(\underline{u}_0, \underline{u}_-)$  the value  $u_0(0)$ , where  $(u_0, u_-)$  is the fixed point of (3.18).

To study the regularity of this map we exploit Lemma 3.2. We set

$$\tilde{X} = \tilde{X}_0 \times \tilde{X}_-,$$

where

$$\tilde{X}_0 = \{u_0 \in C^0([0, +\infty[, \mathbb{R}^{n_0}) : \|u_0\|_0 < +\infty\}$$

and

$$\tilde{X}_- = \{u_- \in C^0([0, +\infty[, \mathbb{R}^{n_-}) : \|u_-\|_- < +\infty\}.$$

Also, let

$$Y = \mathbb{R}^{n_0} \times \mathbb{R}^{n_-}.$$

By direct check one can show that the hypotheses of Lemma 3.2 are all satisfied by the map  $T$  defined by (3.18). In particular,  $T_x$  computed at  $u_0 \equiv \vec{0}$ ,  $u_- \equiv \vec{0}$ ,  $\underline{u}_0 = \vec{0}$  and  $\underline{u}_- = \vec{0}$  is the zero functional. Also, the Fréchet derivative  $T_y$  computed at  $u_0 \equiv \vec{0}$ ,  $u_- \equiv \vec{0}$ ,  $\underline{u}_0 = \vec{0}$  and  $\underline{u}_- = \vec{0}$  is the functional that associates to every  $(h_0, h_-) \in Y$  the function

$$(h_0, e^{\tilde{G}_s \tau} h_-) \in C^0([0, +\infty[, \mathbb{R}^{n_0}) \times C^0([0, +\infty[, \mathbb{R}^{n_-})$$

This implies, in particular, that the map  $\phi_-$  defined by (3.19) is continuously differentiable and that the rows of the jacobian  $D\phi_-(\vec{0}, \vec{0})$  are the vectors  $\vec{e}_1 \dots \vec{e}_{n_0}$ , where  $e_i$  denotes the  $i$ -th element of the canonical basis in  $\mathbb{R}^{n_0+n_-}$ .

We are now ready to define the change of coordinates  $\tilde{\Upsilon}^4$ : if  $U = (\zeta, u_0, u_-)^T$ , then

$$\tilde{\Upsilon}^4(U) = \begin{pmatrix} \zeta \\ \phi_-(u_0, \zeta) \\ u_- \end{pmatrix}.$$

Relying on the previous considerations one can show that the jacobian  $D\tilde{\Upsilon}^4$  is the identity and thus that  $\tilde{\Upsilon}^4$  is locally invertible.

## 4 Invariant manifolds for a singular O.D.E.

The aim of this section is to extend to the general case the considerations introduced in Section 1.1 in the case of a toy model. In particular, in Section 4.1 we extend the definition of fast and slow dynamics, while in Section 4.2 we extend the notion of *generalized stable manifold* and the decomposition formula (1.9).

Even if it not explicitly stated, we always assume that Hypotheses 1 ... 7 hold. Also, we always refer to the formulation

$$\begin{cases} d\zeta/d\tau = G_{10}(\zeta, u_0)u_0\zeta^2 + G_{1-}(\zeta, u_0, u_-)u_-\zeta \\ du_0/d\tau = \left\{ G_{01}(\zeta, u_0) + [G_{0-}(\zeta, u_0, u_-) - G_{0-}(\zeta, u_0, \vec{0})] \right\} \zeta u_0 \\ du_-/d\tau = G_s(\zeta, u_0, u_-)u_- \end{cases} \quad (4.1)$$

## 4.1 Slow and fast dynamics

**Definition 4.1.** *The manifold of slow dynamics is the centre manifold of (4.1), namely the subspace  $\{u_- = \vec{0}\}$ . In the following we denote it by  $\mathcal{M}^0$ .*

*The manifold of fast dynamics of system (4.1) is the subspace  $\{\zeta = 0\}$ .*

Note that both these manifolds are invariant for system (4.1). Also, for every point  $(0, \bar{u}_0, \bar{u}_-)$  belonging to the manifold of fast dynamics, denote by  $(0, u(\tau), u_-(\tau))$  the solution of (4.1) such that

$$(0, u(0), u_-(0)) = (0, \bar{u}_0, \bar{u}_-).$$

Then  $u_0(\tau) \equiv \bar{u}_0$  and  $u_-(\tau)$  decays to zero exponentially fast. Namely,

$$\lim_{\tau \rightarrow +\infty} e^{c\tau/2} |u_-(\tau)| = 0,$$

where the positive constant  $c$  satisfies  $Re\lambda < -c$  for every  $\lambda$  eigenvalue of  $G_s(0, \vec{0}, \vec{0})$ .

Consider system (4.1) reduced on the manifold of slow dynamics:

$$\begin{cases} d\zeta/d\tau = \zeta^2 G_{10}(\zeta, u_0) u_0 \\ du_0/d\tau = G_{01}(\zeta, u_0) u_0 \zeta \\ u_- \equiv 0 \end{cases} \quad (4.2)$$

If one comes back to the original variable  $t$  obtains

$$\begin{cases} d\zeta/dt = \zeta G_{10}(\zeta, u_0) u_0 \\ du_0/dt = G_{01}(\zeta, u_0) u_0 \\ u_- \equiv 0, \end{cases} \quad (4.3)$$

namely an equation with no singularity. Note that (4.2) and (4.3) are equivalent. Indeed, by uniqueness of the solution of a Cauchy problem associated to (4.3), the following holds. If  $u_1(0) > 0$  then  $u_1(t) > 0$  for every  $t$ . Thus, the Cauchy problem

$$\begin{cases} \frac{d\tau}{dt} = \frac{1}{\zeta(t)} \\ \tau(0) = 0 \end{cases} \quad (4.4)$$

admits a global solution  $\tau : ]-\infty, +\infty[ \rightarrow ]-\infty, +\infty[$  whose derivative is always different from 0. Thus,  $\tau(t)$  defines a change of coordinates and (4.2) is equivalent to (4.3).

One of our original goal is to study the solutions of

$$\frac{dU}{dt} = \frac{\phi_s(U)}{\zeta(U)} + \phi_{ns}(U)$$

laying on a centre manifold. Let  $\mathcal{M}^{00}$  be a centre manifold for (4.3) around the equilibrium point  $(0, \vec{0}, \vec{0})$ . Then  $\mathcal{M}^{00}$  is a centre manifold for

$$\begin{cases} \frac{d\zeta}{dt} = \zeta G_{10}(\zeta, u_0) u_0 + G_{1-}(\zeta, u_0, u_-) u_- \\ \frac{du_0}{dt} = \left\{ G_{01}(\zeta, u_0) + [G_{01}(\zeta, u_0, u_-) - G_{0-}(\zeta, u_0, \vec{0})] \right\} u_0 \\ \frac{du_-}{dt} = \frac{1}{\zeta} G_s(\zeta, u_0, u_-) u_- \end{cases} \quad (4.5)$$

We collect these results in the following

**Theorem 4.1.** *Assume that Hypotheses 1 ... 7 are satisfied. There exists an invariant centre manifold  $\mathcal{M}^{00}$  for system (4.5) around the equilibrium point  $(0, \vec{0}, \vec{0})$  which is contained in the manifold of the slow dynamics. In particular, equation (4.5) restricted to  $\mathcal{M}^{00}$  is non singular and every solution satisfies the following property: if  $u_1(0) > 0$ , then  $u_1(t) > 0$  for every  $t$ .*

## 4.2 The generalized stable manifold

The aim of this section is to extend to the general case the definition of *generalized stable manifold* introduced in Section 1.1 for a toy model.

Before stating our results we need to introduce some notations. Consider system (4.1) and let  $V^{0-}$  be the subspace  $\{0, \vec{v}, \vec{0}\}$ , where  $\vec{v}$  belongs to the eigenspace of the matrix  $G_{01}(0, \vec{0})$  associated to eigenvalues with strictly negative real part. We denote by  $n_{0-}$  the dimension of  $V^{0-}$ . Also, let  $V^-$  be the subspace  $\{\zeta = 0, u_0 = \vec{0}\}$ . As before  $E$  denotes the manifold of equilibria  $\{u_0 = \vec{0}, u_- = \vec{0}\}$

Our results is the following

**Theorem 4.2.** *There is a sufficiently small constant  $\delta > 0$  such that the following holds. In the ball of centre  $(0, \vec{0}, \vec{0})$  and of radius  $\delta$  in  $\mathbb{R}^N$  one can define a continuously differentiable manifold  $\mathcal{M}^s$  which has dimension  $(1 + n_{0-} + n_-)$  and satisfies the following properties.*

1. *The manifold  $\mathcal{M}^s$  is parameterized by  $E \oplus V^{0-} \oplus V^-$  and it is tangent to this space at  $(0, \vec{0}, \vec{0})$ .*
2.  *$\mathcal{M}^s$  is invariant with respect to (4.1). Also, if  $\bar{U}(\tau) = (\bar{\zeta}(\tau), \bar{u}_0(\tau), u_-(\tau))$  is an orbit of (4.1) laying on  $\mathcal{M}^s$ , then*

$$\bar{U}(\tau) = U_f(\tau) + U_{sl}(\tau) + U_i(\tau), \quad (4.6)$$

*where  $U_f$  lays on the manifold of fast dynamics and  $U_{sl}$  lays on the manifold of slow dynamics. The function  $U_f(\tau)$  decays exponentially fast to  $(0, \vec{0}, \vec{0})$  as  $\tau \rightarrow +\infty$ . Also, the function  $U_{sl}(\tau(t))$  decays exponentially fast as  $t \rightarrow +\infty$  to an equilibrium in  $E$ . The change of coordinate  $\tau(t)$  is the inverse of the global solution of the Cauchy problem (4.4). The interaction component is small in the sense that*

$$|U_i(\tau)| \leq K|U_f(0)| |U_{sl}(0)| e^{-c\tau/4},$$

*for suitable constants  $K > 0, c > 0$ .*

3. *If the orbit  $\bar{U}(\tau) = (\bar{\zeta}(\tau), \bar{u}_0(\tau), u_-(\tau))$  lays on  $\mathcal{M}^s$  and  $\bar{\zeta}(0) > 0$ , then  $\bar{\zeta}(\tau) > 0$  for every  $\tau$ .*

*More precisely,*

$$|\bar{\zeta}(\tau) - \bar{\zeta}(0)| \leq \tilde{K}\delta|\bar{\zeta}(0)|$$

*for a suitable constant  $\tilde{K}$ .*

We call  $\mathcal{M}^s$  a *generalized stable manifold*. Note that (4.6) may be regarded as an extension of (1.9).

The proof of Theorem 4.2 is given in the remaining part of this section.

## 4.3 Proof of Theorem 4.2

The proof is divided in several steps.

### 4.3.1 First step: further change of coordinates

As a first step, it is convenient to introduce a new change of coordinates. To do this, we consider system (4.1) restricted on the invariant manifold  $\{u_- = \vec{0}\}$ , namely we consider the following equation:

$$\begin{cases} d\zeta/dt = G_{10}(\zeta, u_0)u_0\zeta \\ du_0/dt = G_{01}(\zeta, u_0)u_0. \end{cases} \quad (4.7)$$

Only at the end of Section 4.3.1 we come back to the original system (4.1). Note that we use the variable  $t$ . Thanks to Theorem 4.1, this is equivalent to using the variable  $\tau$ .

The following manifold is entirely constituted by equilibria for (4.7):

$$\mathcal{E} := \{(\zeta, \vec{0}) : \zeta \in \mathbb{R}\}.$$

Let  $M_{\mathcal{E}}^{us}$  be the uniformly stable manifold of (4.7) around the equilibrium point  $(0, \vec{0})$ . Some of the most important properties of the uniformly stable manifold are reviewed in Section 3.0.1. We refer instead to [9] for a complete discussion. Here we just recall that  $M_{\mathcal{E}}^{us}$  is parameterized by  $\mathcal{E} \oplus V^{0-}$ , where  $V^{0-}$  is an eigenspace of the  $(n_0 + 1) \times (n_0 + 1)$  matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & G_{01}(0, \vec{0}) \end{pmatrix}, \quad (4.8)$$

more precisely the eigenspace associated to eigenvalues with strictly negative real part. Modulo a linear change of coordinates, we can assume that

$$\begin{pmatrix} \zeta \\ u_0 \end{pmatrix} = \begin{pmatrix} \zeta \\ u_{0-} \\ u_{0+} \end{pmatrix},$$

and that  $V^{0-} = \{\zeta = 0, u_{0+} = \vec{0}\}$ , while  $\{u_{0-} = \vec{0}\}$  is the eigenspace associated to eigenvalues of (4.8) with non negative real part. We denote by  $n_{0-}$  the dimension of  $V^{0-}$ .

Let

$$\phi_{0-} : \mathcal{E} \times V^{0-} \rightarrow \mathbb{R}^{n_0 - n_{0-}}$$

be a smooth map such that  $(\zeta, u_{0-}, u_{0+})$  belongs to the uniformly stable manifold  $M_{\mathcal{E}}^{us}$  if and only if  $u_{0+} = \phi_{0-}(\zeta, u_{0-})$ . The manifold  $M_{\mathcal{E}}^{us}$  is tangent to  $\mathcal{E} \oplus V^{0-}$  at  $(0, \vec{0}, \vec{0})$  and hence

$$D\phi_{0-}(0, \vec{0}) = \mathbf{0}, \quad (4.9)$$

where  $\mathbf{0}$  is the null  $(n_0 - n_{0-}) \times (1 + n_{0-})$  matrix. Let

$$\tilde{\gamma}_1 : \mathbb{R}^{n_0+1} \rightarrow \mathbb{R}^{n_0+1}$$

be the map defined by

$$\tilde{\gamma}_1 \begin{pmatrix} \zeta \\ u_{0-} \\ u_{0+} \end{pmatrix} = \begin{pmatrix} \zeta \\ u_{0-} \\ u_{0+} - \phi_{0-}(\zeta, u_{0-}) \end{pmatrix}.$$

Because of (4.9), the jacobian  $D\tilde{\gamma}_1(0, \vec{0}, \vec{0})$  is the identity and hence  $\tilde{\gamma}_1$  is locally invertible in a neighbourhood of  $(0, \vec{0}, \vec{0})$ . One can thus proceed as in Section 3.0.2 and show that, if  $(\tilde{\zeta}, \tilde{u}_{0-}, \tilde{u}_{0+})$  denote  $\tilde{\gamma}_1(\zeta, u_{0-}, u_{0+})$ , then in the new coordinates the uniformly stable manifold  $M_{\mathcal{E}}^{us}$  is the subspace  $\{\tilde{u}_{0+} = \vec{0}\}$ . In the following to simplify the notations we write  $(\zeta, u_{0-}, u_{0+})$  instead of  $(\tilde{\zeta}, \tilde{u}_{0-}, \tilde{u}_{0+})$ .

Let  $M^{0+}$  be a centre-unstable manifold for system (4.7) around the equilibrium point

$$(\zeta = 0, u_{0-} = \vec{0}, u_{0+} = \vec{0}).$$

Again we refer to [9] for an extensive discussion about centre-unstable manifolds, here we just recall some of the most important properties. Every centre-unstable manifold is invariant for (4.7) and is parameterized by the eigenspace of (4.8) associated to the eigenvalues with non negative real part. In our case, it is thus parameterized by  $\{u_{0-} = \vec{0}\}$ . Also, every centre-unstable manifold is tangent at  $(\zeta = 0, u_{0-} = \vec{0}, u_{0+} = \vec{0})$  to this eigenspace and satisfies the following property. If  $(\zeta(t), u_{0-}(t), u_{0+}(t))$  is an orbit belonging to the centre-unstable manifold, then

$$\lim_{t \rightarrow -\infty} \left( |z(t)| + |u_{0-}(t)| + |u_{0+}(t)| \right) e^{ct/2} = 0.$$

In the previous formula,  $c$  is a positive constant such that  $\operatorname{Re} \lambda > c$  for every  $\lambda$  eigenvalue of  $G_{01}(0, \vec{0}, \vec{0})$  with strictly positive real part. The parameterization can be chose as follows:  $(\zeta, u_{0-}, u_{0+})$  belong to  $M^{0+}$  if and only if  $u_{0-} = \phi_{0+}(\zeta, u_{0+})$  for a suitable smooth map  $\phi_{0+}$ . Proceeding as before, one can define a smooth and locally invertible change of coordinates  $\tilde{\gamma}_2$  such that the following holds. Let  $(\tilde{\zeta}, \bar{u}_{0-}, u_{0+}) = \tilde{\gamma}_2(\zeta, u_{0-}, u_{0+})$ , then the centre-unstable manifold  $M^{0+}$  is in the new coordinates the subspace  $\{\bar{u}_{0-} = \vec{0}\}$ , while the uniformly stable manifold  $M_{\mathcal{E}}^{us}$  is  $\{\bar{u}_{0+} = \vec{0}\}$ . To simplify the notations in the following we write  $(\zeta, u_{0-}, u_{0+})$  instead of  $(\tilde{\zeta}, \bar{u}_{0-}, u_{0+})$ .

Exploiting the invariance of the manifolds  $M^{0+}$  and  $M_{\mathcal{E}}^{us}$  and the regularity of the maps  $G_{01}$  and  $G_{10}$ , we get that (4.7) written in the new coordinates takes the form

$$\begin{cases} d\zeta/dt = G_{10-}(\zeta, u_{0-}, u_{0+})u_{0-}\zeta + G_{10+}(\zeta, u_{0-}, u_{0+})u_{0+}\zeta \\ du_{0-}/dt = G_{00-}(\zeta, u_{0-}, u_{0+})u_{0-} \\ du_{0+}/dt = G_{00+}(\zeta, u_{0-}, u_{0+})u_{0+}. \end{cases} \quad (4.10)$$

In the previous expression,  $G_{00-} \in \mathbb{M}^{n_{0-} \times n_{0-}}$ ,  $G_{00+} \in \mathbb{M}^{(n_0 - n_{0-}) \times (n_0 - n_{0-})}$  and the row vectors  $G_{10-}$  and  $G_{10+}$  belong to  $\mathbb{R}^{n_{0-}}$  and to  $\mathbb{R}^{n_0 - n_{0-}}$  respectively. By construction,  $G_{00-}$  admits only eigenvalues with strictly negative real part.

Let  $\tilde{\Upsilon}^7 : \mathbb{R}^N \rightarrow \mathbb{R}^N$  be the map defined by

$$\tilde{\Upsilon}^7 \begin{pmatrix} \zeta \\ u_{0-} \\ u_{0+} \\ u_- \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_2 \circ \tilde{\gamma}_1(\zeta, u_{0-}, u_{0+}) \\ u_- \end{pmatrix}.$$

Thanks to the previous considerations the jacobian  $D\tilde{\Upsilon}^7$  is an invertible matrix, hence  $\Upsilon^7$  is locally invertible and we can proceed as in Section 3.0.2. System (4.1) written in the new coordinates takes the form

$$\begin{cases} d\zeta/d\tau = G_{10-}(\zeta, u_{0-}, u_{0+})u_{0-}\zeta^2 + G_{10+}(\zeta, u_{0-}, u_{0+})u_{0+}\zeta^2 + G_{1-}(\zeta, u_{0-}, u_{0+}, u_-)u_- \zeta \\ du_{0-}/d\tau = G_{00-}(\zeta, u_{0-}, u_{0+})u_{0-}\zeta + [G_{0--}(\zeta, u_{0-}, u_{0+}, u_-) - G_{0--}(\zeta, u_{0-}, u_{0+}, \vec{0})]u_0\zeta \\ du_{0+}/d\tau = G_{00+}(\zeta, u_{0-}, u_{0+})u_{0+}\zeta + [G_{0-+}(\zeta, u_{0-}, u_{0+}, u_-) - G_{0-+}(\zeta, u_{0-}, u_{0+}, \vec{0})]u_0\zeta \\ du_-/d\tau = G_s(\zeta, u_{0-}, u_{0+}, u_-)u_-, \end{cases} \quad (4.11)$$

where the matrices  $G_{0--}$  and  $G_{0-+}$  belong respectively to  $\mathbb{M}^{n_{0-} \times n_0}$  and to  $\mathbb{M}^{(n_0 - n_{0-}) \times n_0}$ . The functions  $G_{10-}$ ,  $G_{10+}$ ,  $G_{00-}$  and  $G_{00+}$  are as before.

#### 4.3.2 Second step: analysis of the fast dynamic

Fix  $\underline{u}_{0-} \in \mathbb{R}^{n_{0-}}$  such that  $|\underline{u}_{0-}| \leq \delta$  and consider the equation

$$\begin{cases} \zeta(\tau) \equiv 0 \\ u_{0-}(\tau) \equiv \underline{u}_{0-} \\ u_{0+}(\tau) \equiv 0 \\ du_-/d\tau = G_s(\vec{0}, \underline{u}_{0-}, \vec{0}, u_-)u_- \end{cases} \quad (4.12)$$

Every solution of (4.12) lays on the manifold of fast dynamics.

Let

$$X^- := \left\{ u_- \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^{n_-}) : \|u_-\|_- \leq k_- \delta \right\},$$

where the norm  $\|\cdot\|_-$  is defined by

$$\|u_-\|_- = \sup_{\tau} \{|u_-(\tau)|e^{c\tau/2}\}.$$



As usual,  $c$  denotes a positive constant such that  $\operatorname{Re} \lambda < -c$  for every  $\lambda$  eigenvalue of  $G_s(0, \vec{0}, \vec{0}, \vec{0})$  with strictly negative real part. We specify in the following how to assign the exact value of the constant  $k_-$ . Also, let  $Y^-$  denote the ball of centre  $(\vec{0}, \vec{0})$  and radius  $\delta$  in  $\mathbb{R}^{n_{0-}} \times \mathbb{R}^{n_-}$ .

Fix  $\underline{u}_- \in \mathbb{R}^{n_-}$  such that  $|\underline{u}_-| \leq \delta$ . The solution of (4.12) satisfying  $u_-(0) = \underline{u}_-$  is the fixed point of the map  $T^-$  with domain  $X \times Y^-$  and defined by

$$T^-(u_-, \underline{u}_{0-}, \underline{u}_-)(\tau) = e^{\bar{G}_s \tau} \underline{u}_- + \int_0^\tau e^{\bar{G}_s(\tau-s)} [G_s(\vec{0}, \underline{u}_{0-}, \vec{0}, u_-) - \bar{G}_s] u_-(s) ds. \quad (4.13)$$

In the previous expression,  $\bar{G}_s = G_s(0, \vec{0}, \vec{0}, \vec{0})$ .

If  $\delta$  is small enough, then the constant  $k_-$  can be chosen in such a way that, for every fixed  $(\underline{u}_{0-}, \underline{u}_-)$  in  $Y^-$ , the map  $T^-(\cdot, \underline{u}_{0-}, \underline{u}_-)$  is a contraction with Lipschitz constant less or equal to  $1/2$ . As pointed out in Section 3.0.1, in this way we define a function which associates to every couple  $(\underline{u}_{0-}, \underline{u}_-)$  the fixed point of the map  $T^-(\cdot, \underline{u}_{0-}, \underline{u}_-)$ . Relying on Lemma 3.2, one gets that such a function is continuously differentiable and that the Frechét derivative at the point  $(\underline{u}_{0-} = \vec{0}, \underline{u}_- = \vec{0})$  is the functional

$$(h_-, h_0) \mapsto e^{\bar{G}_s \tau} h_-.$$

Also, the fixed point of the map  $T^-(\cdot, (\underline{u}_{0-}, \underline{u}_-))$  satisfies

$$|u_-(\tau)| \leq e^{-c\tau/2} |\underline{u}_-| \quad (4.14)$$

### 4.3.3 Third step: analysis of the slow dynamic

Consider the equation

$$\begin{cases} d\zeta/dt = G_{10-}(\zeta, u_{0-}, \vec{0}) u_{0-} \zeta \\ du_{0-}/dt = G_{00-}(\zeta, u_{0-}, \vec{0}) u_{0-} \\ u_{0+}(t) \equiv \vec{0} \\ u_-(t) \equiv \vec{0} \end{cases} \quad (4.15)$$

Every solution of (4.15) lays on the manifold of the slow dynamics and hence it is equivalent to use the variable  $t$  or the variable  $\tau$ .

Let  $Y^0$  denote the ball of centre  $(0, \vec{0})$  and radius  $\delta$  in  $\mathbb{R} \times \mathbb{R}^{n_{0-}}$ . Also, let

$$X^\zeta := \{\zeta \in \mathcal{C}^0([0, +\infty[) : \forall t, |\zeta(t)| \leq 2|\zeta(0)|, |z(t)| \leq k_\zeta \delta\}$$

and

$$X^{0-} := \{u_{0-} \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^{n_{0-}}) : \|u_{0-}\|_{0-} \leq k_{0-} \delta\}.$$

The norm  $\|\cdot\|_{0-}$  is defined as follows:

$$\|u_{0-}\|_{0-} = \sup_t \{|u_{0-}(t)| e^{ct/2}\},$$

where the positive constant  $c$  satisfies  $\operatorname{Re} \lambda < -c$  for every  $\lambda$  eigenvalue of  $G_{00-}(0, \vec{0}, \vec{0})$ . The space  $X^\zeta$  is equipped with the standard  $\mathcal{C}^0$  norm. We specify in the following how to determine the constants  $k_\zeta$  and  $k_{0-}$ . Given  $(\zeta, \underline{u}_{0-}) \in Y^0$ , the solution of (4.15) satisfying  $\zeta(0) = \zeta$  and  $u_{0-}(0) = \underline{u}_{0-}$  is a fixed point of the map  $T^0(\cdot, \zeta, \underline{u}_{0-})$ , defined on the domain  $(\zeta, u_{0-}) \in X^\zeta \times X^{0-}$  as follows:

$$\begin{cases} T_1^0(\zeta, u_{0-})[t] = \zeta + \int_0^t G_{10-}(\zeta(s), u_{0-}(s), \vec{0}) u_{0-}(s) \zeta(s) ds \\ T_2^0(\zeta, u_{0-})[t] = e^{\bar{G}_{00-} t} \underline{u}_{0-} + \int_0^t e^{\bar{G}_{00-}(t-s)} [G_{00-}(\zeta(s), u_{0-}(s), \vec{0}) - \bar{G}_{00-}] u_{0-}(s) ds \end{cases} \quad (4.16)$$

In the previous expression,  $\bar{G}_{00-} = G_{00-}(0, \vec{0}, \vec{0})$ .

If  $\delta$  is sufficiently small, then the constants  $k_\zeta$  and  $k_{0-}$  can be chosen in such a way that the following property holds. For every  $(\underline{\zeta}, \underline{u}_{0-}) \in Y^0$ , the map  $T^0(\cdot, \underline{\zeta}, \underline{u}_{0-})$  is a contraction on  $X^\zeta \times X^{0-}$  and the Lipschitz constant is less or equal to  $1/2$ .

One can then proceed as in Section 4.3.2 and define a continuously differentiable map

$$Y^0 \rightarrow X^0$$

which associates to every  $(\underline{\zeta}, \underline{u}_{0-}) \in Y^0$  the fixed point of the map  $T^0(\cdot, \underline{\zeta}, \underline{u}_{0-})$ . The Frechét derivative at  $(\underline{\zeta} = 0, \underline{u}_{0-} = \vec{0})$  is the functional

$$(h_\zeta, h_{0-}) \mapsto \begin{pmatrix} h_\zeta \\ e^{\bar{G}_{00-}t} h_{0-} \end{pmatrix}.$$

Note that *a posteriori* the fixed point of the map  $T^0(\cdot, \underline{\zeta}, \underline{u}_{0-})$  satisfies

$$|u_{0-}(t)| \leq e^{-ct/2} |\underline{u}_{0-}| \quad |\zeta(t) - \underline{\zeta}| \leq 2M |\underline{u}_{0-}| |\underline{\zeta}|,$$

where  $M = \max_{Y^0} \{G_{10-}\} 4k_{0-}/c$ . In particular, this implies that the solutions of

$$\begin{cases} d\zeta/d\tau = G_{10-}(\zeta, u_{0-}, \vec{0}) u_{0-} \zeta^2 \\ du_{0-}/d\tau = G_{00-}(\zeta, u_{0-}, \vec{0}) u_{0-} \zeta \\ u_{0+}(\tau) \equiv \vec{0} \\ u_{-}(\tau) \equiv \vec{0} \end{cases}$$

satisfying  $u_{0-}(0) = \underline{u}_{0-}$  and  $\zeta(0) = \underline{\zeta}(0)$  verify the following estimate:

$$|u_{0-}(\tau)| \leq |\underline{u}_{0-}| \quad |\zeta(\tau) - \underline{\zeta}| \leq 2M |\underline{u}_{0-}| |\underline{\zeta}|. \quad (4.17)$$

Also, since

$$u_{0-}(\tau) - u_{0-} = \int_0^\tau G_{00-}(\zeta(s), u_{0-}(s), \vec{0}) u_{0-}(s) \zeta(s) ds$$

from (4.17) one gets

$$|u_{0-}(\tau) - \underline{u}_{0-}| \leq \tilde{M} \tau |\underline{u}_{0-}| |\underline{\zeta}|, \quad (4.18)$$

where  $\tilde{M} = 2 \max_{Y^0} \{G_{00-}\}$

#### 4.3.4 Step four: the analysis of the component of interaction

Fix values  $(\underline{\zeta}, \underline{u}_{0-}, \underline{u}_{-}) \in \mathbb{R} \times \mathbb{R}^{n_0} \times \mathbb{R}^{n-}$ . Let  $u_{-}(\tau)$  be the fixed point of the map  $T^-$  defined by (4.13) and let  $(\zeta(t), u_{0-}(t))$  be the fixed of  $T^0$ , the map defined by (4.16). The aim of this section is to study the component of interaction

$$(z(\tau), p(\tau), q(\tau), w(\tau)).$$

In the previous expression,  $z(\tau) \in \mathbb{R}$ ,  $p(\tau) \in \mathbb{R}^{n_0-}$ ,  $q(\tau) \in \mathbb{R}^{n_0-n_0-}$  and  $w(\tau) \in \mathbb{R}^{n-}$ . These components are defined in such a way that

$$\bar{U}(\tau) = \begin{pmatrix} \zeta(t(\tau)) \\ u_{0-}(t(\tau)) \\ \vec{0} \\ \vec{0} \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{0} \\ \vec{0} \\ u_{-}(\tau) \end{pmatrix} + \begin{pmatrix} z(\tau) \\ p(\tau) \\ q(\tau) \\ w(\tau) \end{pmatrix} \quad (4.19)$$

is a solution of (4.11), namely

$$\begin{cases} d\bar{\zeta}/d\tau = G_{10-}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}) \bar{u}_{0-} \bar{\zeta}^2 + G_{10+}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}) \bar{u}_{0+} \bar{\zeta}^2 + G_{1-}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}, \bar{u}_{-}) \bar{u}_{-} \bar{\zeta} \\ d\bar{u}_{0-}/d\tau = G_{00-}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}) \bar{u}_{0-} \bar{\zeta} + [G_{0--}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}, \bar{u}_{-}) - G_{0--}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}, \vec{0})] \bar{u}_{0-} \bar{\zeta} \\ d\bar{u}_{0+}/d\tau = G_{00+}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}) \bar{u}_{0+} \bar{\zeta} + [G_{0-+}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}, \bar{u}_{-}) - G_{0-+}(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}, \vec{0})] \bar{u}_{0+} \bar{\zeta} \\ d\bar{u}_{-}/d\tau = G_s(\bar{\zeta}, \bar{u}_{0-}, \bar{u}_{0+}, \bar{u}_{-}) \bar{u}_{-}. \end{cases}$$

In formula (4.19), the function  $t(\tau)$  is the inverse of the function  $\tau(t)$  which is the maximal (and global) solution of the Cauchy problem (4.4),

$$\begin{cases} d\tau/dt = 1/\zeta(t) \\ \tau(0) = 0, \end{cases}$$

In the following to simplify the notations we just write  $\zeta(\tau)$  and  $u_{0-}(\tau)$  instead of  $\zeta(t(\tau))$  and  $u_{0-}(t(\tau))$ .

We also require that the component of interaction is exponentially decaying to an equilibrium in the subspace  $\{u_{0-} = \vec{0}, u_{0+} = \vec{0}, u_- = \vec{0}\}$ . More precisely, we define (4.19) as the fixed point of the map  $T^i$  whose component are defined as follows.

$$\begin{aligned} T_1^i(z, p, q, w, \underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)[\tau] &= \int_{+\infty}^{\tau} \left[ G_{10-}(\zeta + z, u_{0-} + p, q) - G_{10-}(\zeta, u_{0-}, \vec{0}) \right] u_{0-} \zeta^2 ds \\ &+ \int_{+\infty}^{\tau} G_{10-}(\zeta + z, u_{0-} + p, q) \left[ \zeta^2 p + (z^2 + 2\zeta z)(u_{0-} + p) \right] ds + \int_{+\infty}^{\tau} G_{10+}(\zeta + z, u_{0-} + p, q) (\zeta + z) q ds \\ &+ \int_{+\infty}^{\tau} G_{1-}(\zeta + z, u_{0-} + p, q, u_- + w) (u_- + w) (\zeta + z) ds \\ T_2^i(z, p, q, w, \underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)[\tau] &= \int_{+\infty}^{\tau} \left[ G_{00-}(\zeta + z, u_{0-} + p, q) - G_{00-}(\zeta, u_{0-}, \vec{0}) \right] u_{0-} \zeta ds \\ &+ \int_{+\infty}^{\tau} G_{00-}(\zeta + z, u_{0-} + p, q) \left[ z u_{0-} + (\zeta + z) p \right] ds \\ &+ \int_{+\infty}^{\tau} \left[ G_{0--}(\zeta + z, u_{0-} + p, q, u_- + w) - G_{0--}(\zeta + z, u_{0-} + p, q, \vec{0}) \right] (u_{0-} + p, q)^T (\zeta + z) ds \end{aligned}$$

In the previous expression,  $(u_{0-} + p, q)^T$  stands for the vector

$$\begin{pmatrix} u_{0-} + p \\ q \end{pmatrix}$$

The last two components of  $T$  are defined as follows:

$$\begin{aligned} T_3^i(z, p, q, w, \underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)[\tau] &= \int_{+\infty}^{\tau} G_{00+}(\zeta + z, u_{0-} + p, q) q (\zeta + z) ds \\ &+ \int_{+\infty}^{\tau} \left[ G_{0-+}(\zeta + z, \bar{u}_{0-} + p, q, u_- + w) - G_{0-+}(\zeta + z, \bar{u}_{0-} + p, q, \vec{0}) \right] (u_{0-} + p, q)^T (\zeta + z) ds \\ T_4^i(z, p, q, w, \underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)[\tau] &= \int_0^{\tau} e^{\bar{G}_s(\tau-s)} \left[ \left( G_s(\zeta + z, u_{0-} + p, q, u_- + w) - \bar{G}_s \right) w \right. \\ &\quad \left. + \left( G_s(\zeta + z, \bar{u}_{0-} + p, q, u_- + w) - G_s(0, \underline{u}_{0-}, \vec{0}, u_-) \right) u_- \right] ds \end{aligned}$$

In the definition of  $T_4^i$  we use the notation  $\bar{G}_s = G_s(0, \vec{0}, \vec{0}, \vec{0})$ .

Consider the spaces

$$\begin{aligned} X_1^i &:= \left\{ z \in \mathcal{C}^0([0, +\infty[) : \|z\|_{i1} \leq c_{i1} |\underline{\zeta}| |\underline{u}_-| \right\} & X_2^i &:= \left\{ p \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^{n_{0-}}) : \|p\|_{i2} \leq c_{i2} |\underline{\zeta}| |\underline{u}_-| \right\} \\ X_3^i &:= \left\{ q \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^{n_0 - n_{0-}}) : \|q\|_{i3} \leq c_{i3} |\underline{\zeta}| |\underline{u}_-| \right\} & X_4^i &:= \left\{ w \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^{n-}) : \|w\|_{i4} \leq c_{i4} |\underline{\zeta}| |\underline{u}_-| \right\}, \end{aligned} \tag{4.20}$$

where the norms  $\|\cdot\|_{i1}, \dots, \|\cdot\|_{i4}$  are defined as follows:

$$\begin{aligned} \|z\|_{i1} &= \sup_{\tau} \{ e^{c\tau/4} |z(\tau)| \} & \|p\|_{i2} &= \sup_{\tau} \{ e^{c\tau/4} |p(\tau)| \} \\ \|q\|_{i3} &= \sup_{\tau} \{ e^{c\tau/4} |q(\tau)| \} & \|w\|_{i4} &= \sup_{\tau} \{ e^{c\tau/4} |w(\tau)| \}. \end{aligned}$$

In the previous expression, the positive constant  $c$  satisfies  $\operatorname{Re} \lambda < -c$  for every  $\lambda$  eigenvalue of  $G_s(0, \vec{0}, \vec{0}, \vec{0})$  and for every  $\lambda$  eigenvalue of  $G_{00-}(0, \vec{0}, \vec{0}, \vec{0})$ .

The values  $\underline{\zeta}$ ,  $\underline{u}_{0-}$  and  $\underline{u}_-$  satisfy  $|\underline{\zeta}|, |\underline{u}_{0-}|, |\underline{u}_-| \leq \delta$ . By direct check, one can show that, if  $\delta$  is small enough, then the constants  $c_{i1}, \dots, c_{i4}$  can be chosen in such a way that the map  $T^i$  is a contraction on the space  $X_1^i \times X_2^i \times X_3^i \times X_4^i$ . The Lipschitz constant is less or equal to  $1/2$ .

In this way, we define a function on the ball of radius  $\delta$  an centre  $(0, \vec{0}, \operatorname{vec} 0)$  in  $\mathbb{R} \times \mathbb{R}^{n_{0-}} \times \mathbb{R}^{n_-}$ . This function takes values in  $X_1^i \times X_2^i \times X_3^i \times X_4^i$  and associates to  $(\underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)$  the fixed point of the map  $T^i$ . Concerning the regularity of this function, we can proceed as follows.

As pointed out in Section 4.3.3, there is also a map defined on the ball of radius  $\delta$  and centre  $\vec{0}$  in  $\mathbb{R} \times \mathbb{R}^{n_{0-}}$ . This map takes values in  $X^\zeta \times X^{0-}$  and associates to  $(\underline{\zeta}, \underline{u}_{0-})$  the fixed point of (4.16) is continuously differentiable. Thus, the same map is continuously differentiable if regarded as a map taking values in  $\mathcal{C}^0([0, +\infty[, \mathbb{R}) \times \mathcal{C}^0([0, +\infty[, \mathbb{R}^{n_{0-}})$  equipped with the standard uniform norm. Indeed, this norm is weaker than the norm on  $X^\zeta \times X^{0-}$ .

Also, the map

$$\mathbb{R}^{n_-} \rightarrow X^-$$

that associates to  $\underline{u}_-$  the fixed point of the map (4.13) is continuously differentiable.

By direct check, it turns out that for every fixed  $(z, p, q, w)$  the map  $T^i$  is continuously differentiable with respect to  $(\underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)$ . To give a flavour of the proof of the differentiability we fix  $h \in \mathbb{R}$  and we consider the term

$$\begin{aligned} & \int_{+\infty}^{\tau} \left[ G_{10-}(\zeta(\underline{\zeta} + h) + z, u_{0-} + p, q) - G_{10-}(\zeta(\underline{\zeta} + h), u_{0-}, \vec{0}) \right] u_{0-} \zeta^2 ds \\ & - \int_{+\infty}^{\tau} \left[ G_{10-}(\zeta(\underline{\zeta} + h) + z, u_{0-} + p, q) - G_{10-}(\zeta(\underline{\zeta} + h), u_{0-}, \vec{0}) \right] u_{0-} \zeta^2 ds \\ & = \int_{+\infty}^{\tau} \left[ \int_0^1 \left[ D_q G_{10-}(\zeta(\underline{\zeta} + h) + z, u_{0-} + p, rq) - D_q G_{10-}(\zeta(\underline{\zeta}) + z, u_{0-} + p, rq) \right] q(s) dr \right] u_{0-} \zeta^2 ds = \\ & = \int_{+\infty}^{\tau} \left[ \int_0^1 \left[ L(\zeta(\underline{\zeta}) + z, u_{0-} + p, rq)[h] + o(|h|) \right] q(s) u_{0-} \zeta^2 ds \right] \\ & = \mathcal{L}(z, p, q, w, \underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)[h] + o(|h|) \end{aligned}$$

In the previous expression, when it is not otherwise explicitly stated  $\zeta$  denotes  $\zeta(\underline{\zeta})$ . The symbols  $L$  and  $\mathcal{L}$  denote two linear operators both defined on  $\mathbb{R}$  taking values in  $\mathbb{M}^{n_{0-} \times n_-}$  and in  $X_1^i$  respectively. They also depend on  $(z, p, q, w, \underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)$ . The other terms in the difference

$$T^i(z, p, q, w, \underline{\zeta} + h, \underline{u}_{0-} + k, \underline{u}_- + j) - T^i(z, p, q, w, \underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)$$

can be handled in a similar way for every  $h \in \mathbb{R}$ ,  $k \in \mathbb{R}^{n_{0-}}$  and  $j \in \mathbb{R}^{n_-}$ . In this way, we define the Fréchet derivative  $T_y^i$ , which satisfies all the hypotheses in Lemma 3.2. Also, one can show the existence of the derivative Fréchet derivative  $T_x^i$  which continuously differentiable in the sense specified in the statement of Lemma 3.2. The map that associates to  $(\underline{\zeta}, \underline{u}_{0-}, \underline{u}_-)$  the fixed point of the map  $T^i$  is thus continuously differentiable. Also, from the definitions (4.20) it follows

$$\|z\|_{i1} \leq c_{i1} |\underline{\zeta}| |\underline{u}_-| \quad \|p\|_{i2} \leq c_{i2} |\underline{\zeta}| |\underline{u}_-| \quad \|q\|_{i3} \leq c_{i3} |\underline{\zeta}| |\underline{u}_-| \quad \|w\|_{i4} \leq c_{i4} |\underline{\zeta}| |\underline{u}_-|$$

and hence that the Fréchet derivative at  $(\underline{\zeta} = 0, \underline{u}_{0-} = \vec{0}, \underline{u}_- = \vec{0})$  is the zero operator.

### 4.3.5 Conclusion

The map  $\phi_s$  that parameterizes the generalized stable manifold can be defined as follows:

$$\phi_s(\underline{\zeta}, \underline{u}_{0-}, \underline{u}_{-}) = \begin{pmatrix} \underline{\zeta} \\ \underline{u}_{0-} \\ \vec{0} \\ \vec{0} \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{0} \\ \vec{0} \\ \underline{u}_{-} \end{pmatrix} + \begin{pmatrix} z(0) \\ p(0) \\ q(0) \\ 0 \end{pmatrix},$$

where  $(z, p, q, w)$  is the fixed point of the map  $T^i$  defined in Section 4.3.4. Thanks to the considerations made at the end of Section 4.3.4, the function  $\phi_s$  is continuously differentiable and the columns of the jacobian  $D\phi_s(0, \vec{0}, \vec{0})$  are a basis for the subspace  $\{u_{0+} = \vec{0}\}$ . This concludes the proof of Theorem 4.2.

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